Articulated Course

in

Mathematics

V

Mathematical Physics

From Symmetries to Strings

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TO THE READER

This course articulates in five volumes and over twelve hundred articles the grammar by which reality composes itself, from the first whole number to the Langlands correspondence, from the brachistochrone to the amplituhedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.

Each article develops a single idea, from its motivation to its interpretation. Definitions are framed; theorems are also framed, in red. Proofs are present where they illuminate, sketched where full rigour would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The reader will have no trouble distinguishing the three cases.

Ten mathematical objects traverse the collection like red threads: the circle S^1*, the integers* $\mathbb{Z}$*, the extension* $\mathbb{Q}(\sqrt{2})$*, the elliptic curve* $y^2 = x^3 - x$*, the group* GL_2*, the function* $\zeta(s)$*, the space* L^2*, the ring* $K[X]$*, the symmetric group* $\mathfrak{S}_3$*, and the torus* T^2*. From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.*

The figures, numbering six hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colours. They do not replace demonstration; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.

The reader may follow the linear path or take the transversal passages between volumes. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands programme; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.

Every error reported is an error corrected.

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CHAPTER I

LAGRANGIAN MECHANICS

The motion of a mechanical system renders stationary a functional, the action. This variational principle, due to Hamilton, unifies all of classical mechanics in a single idea. The Euler-Lagrange equations and Noether's theorem are its immediate consequences.

Articles in this chapter:

001. *Principle of least action — Motion renders the action stationary; this variational principle unifies mechanics.*
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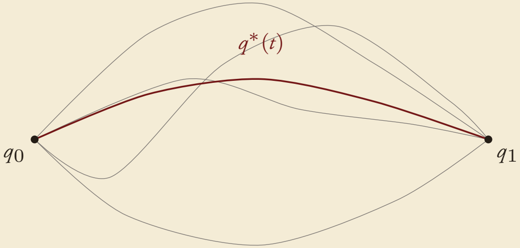
001. PRINCIPLE OF LEAST ACTION.

WHY, among all the trajectories a body could follow between two positions, does nature select exactly one? Newton answers with forces; Lagrange, with a principle of an altogether different scope. The realised motion is the one that renders stationary a certain global quantity, the *action*, computed over the entire trajectory. This reversal of perspective, from the local to the global, is comparable to the shift from a grammar that corrects word by word to one that judges the sentence as a whole.

DEFINITION 001.1: Let Q be a configuration manifold of dimension n, and let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a smooth function called the *Lagrangian*. For a curve $q : [t_0, t_1] \to Q$ of class C^2, the *action functional* is

$$S[q] = \int_{t_0}^{t_1} L\big(q(t),\, \dot{q}(t),\, t\big)\, dt.$$



$$\delta S[q^*] = 0$$

Among the paths connecting q_0 to q_1, the physical trajectory $q^*(t)$ renders the action stationary.

The action is not a number attached to an instant; it is a number attached to a path. All of Lagrangian mechanics rests on the idea that this number, among all paths joining two given configurations, reaches a critical point on the physical trajectory.

THÉORÈME (HAMILTON'S PRINCIPLE): Let q^* be the physical trajectory of a mechanical system with Lagrangian L, connecting configuration q_0 at time t_0 to configuration q_1 at time t_1. Then q^* is a critical point of the functional S among all C^2 curves satisfying $q(t_0) = q_0$ and $q(t_1) = q_1$:

$$\delta S[q^*] = 0,$$

where δS denotes the first variation of S.

The condition $\delta S = 0$ does not mean that the action is minimal. For short times, the physical trajectory often minimises the action; for long times, it may correspond to a saddle point. The term ”least action,” inherited from Maupertuis, is therefore a misnomer sanctioned by tradition.

REMARQUE : The Lagrangian of a classical mechanical system takes the form $L = T - V$, the difference between kinetic energy T and potential energy V. This choice is not arbitrary: it is the only one, up to a total divergence, that reproduces Newton’s equations in arbitrary coordinates. The asymmetry $T - V$ (rather than $T + V$) reflects the competition between inertia and interaction, and it is this tension that the action integrates over time.

EXAMPLE : For a particle of mass m in a potential $V(q)$ *in one dimension, the Lagrangian is* $L(q, \dot{q}) = \frac{1}{2} m\dot{q}^2 - V(q)$*. The action* $S[q] = \int_{t_0}^{t_1} \left(\frac{1}{2} m\dot{q}^2 - V(q)\right) dt$ *is stationary on the Newtonian trajectory* $m\ddot{q} = -V'(q)$.

The power of Hamilton’s principle lies not in the equations it produces (they are Newton’s) but in the freedom it offers: the Lagrangian transforms transparently under a change of coordinates, and every symmetry of the Lagrangian will translate, via Noether’s theorem, into a conservation law. It is this variational architecture that, beyond mechanics, will structure field theory and quantum physics.

002. EULER-LAGRANGE EQUATIONS.

HAMILTON’S principle asserts that the physical trajectory renders the action stationary; what remains is to translate this global condition into a local equation. The passage from one to the other is the founding act of the calculus of variations: an integral condition on an entire path condenses into a differential equation that the trajectory must satisfy at every instant. It is as though a requirement bearing on the coherence of a whole sentence reduced to a grammatical rule applicable word by word.

THÉORÈME (EULER-LAGRANGE): Let $L(q, \dot{q}, t)$ be a C^2 Lagrangian and $q^* : [t_0, t_1] \to \mathbb{R}^n$ a critical point of the action functional $S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt$ among C^2 curves with fixed endpoints. Then q^* satisfies the *Euler-Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, \ldots, n.$$

PREUVE: Let $\eta : [t_0, t_1] \to \mathbb{R}^n$ be a C^2 variation with $\eta(t_0) = \eta(t_1) = 0$. Set $q_\varepsilon = q^* + \varepsilon\eta$. Stationarity of S gives

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S[q_\varepsilon] = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i}\eta^i + \frac{\partial L}{\partial \dot{q}^i}\dot{\eta}^i \right) dt.$$

Integrating the second term by parts and using $\eta(t_0) = \eta(t_1) = 0$ yields

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right) \eta^i \, dt.$$

Since η is arbitrary, the Du Bois-Reymond lemma forces the integrand to vanish identically.

These equations form a system of n second-order ordinary differential equations. When the Lagrangian is $L = \frac{1}{2}m|\dot{q}|^2 - V(q)$, one recovers $m\ddot{q}^i = -\partial V/\partial q^i$: Newton's equations. The virtue of the Euler-Lagrange formulation is that it presupposes no privileged coordinate system.

PROPOSITION: The Euler-Lagrange equations are covariant: if $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $\tilde{q} = Q(q)$, then the Lagrangian $\tilde{L}(\tilde{q}, \dot{\tilde{q}}, t) = L(Q^{-1}(\tilde{q}), DQ^{-1} \cdot \dot{\tilde{q}}, t)$ produces the same physical trajectories expressed in the new coordinates.

This covariance is automatic: it follows from the fact that the variational principle involves only the value of the integral $S[q]$, which is a scalar. Newton's laws, written in Cartesian coordinates, must be corrected by inertial terms in curvilinear coordinates; the Euler-Lagrange equations adapt without correction.

EXAMPLE: *In polar coordinates* (r,θ), *the Lagrangian of a free particle of mass* m *is* $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. *The Euler-Lagrange equations give* $m\ddot{r} - mr\dot{\theta}^2 = 0$ *and* $\frac{d}{dt}(mr^2\dot{\theta}) = 0$. *The centrifugal term* $mr\dot{\theta}^2$ *and the conservation of angular momentum* $mr^2\dot{\theta}$ *appear without any vectorial reasoning.*

REMARQUE: The quantity $p_i = \partial L/\partial \dot{q}^i$ is the *conjugate momentum* to q^i. If L does not depend explicitly on q^j, the Euler-Lagrange equation for that coordinate reduces to $\dot{p}_j = 0$: the conjugate momentum is conserved. This link between the ignorability of a coordinate and the conservation of a momentum is the simplest instance of Noether's theorem.

The Euler-Lagrange equations thus constitute the local translation of a global principle. Their structure, relating partial derivatives of the Lagrangian to time derivatives of the momenta, contains in germ the transition to the Hamiltonian formalism.

003. SYMMETRIES AND NOETHER'S THEOREM.

ENERGY is conserved, momentum is conserved, angular momentum is conserved. Three laws, learned separately, verified case by case. Is there a single principle that generates them all? Noether's theorem answers: every conservation law is the shadow of a continuous symmetry of the Lagrangian. Time-translation invariance produces energy; space-translation invariance produces momentum; rotational invariance produces angular momentum. This theorem weaves so tight a link between symmetry and conservation that one cannot exist without the other.

DEFINITION 003.1: Let $L(q,\dot{q},t)$ be a Lagrangian. A one-parameter group of diffeomorphisms $\phi_s : Q \to Q$, $s \in \mathbb{R}$, with $\phi_0 = \mathrm{id}$, is a *symmetry of the Lagrangian* if

$$L\big(\phi_s(q),\, D\phi_s \cdot \dot{q},\, t\big) = L(q,\dot{q},t)$$

for all s, or more generally if the difference is a total time derivative dF_s/dt.

THÉORÈME (NOETHER) : Let ϕ_s be a one-parameter symmetry of the Lagrangian L, with infinitesimal generator $\delta q^i = \frac{d}{ds}\big|_{s=0} \phi_s(q)^i$. Then the quantity

$$I = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i}\, \delta q^i$$

is conserved along every solution of the Euler-Lagrange equations: $\dot{I} = 0$.

PREUVE : Differentiate $L(\phi_s(q), D\phi_s \cdot \dot{q}, t) = L(q, \dot{q}, t)$ with respect to s at $s = 0$:

$$\frac{\partial L}{\partial q^i}\, \delta q^i + \frac{\partial L}{\partial \dot{q}^i}\, \delta \dot{q}^i = 0.$$

Using the Euler-Lagrange equations $\frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}$ and the fact that $\delta \dot{q}^i = \frac{d}{dt} \delta q^i$, the left-hand side rewrites as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i}\, \delta q^i \right) = 0.$$

The quantity in parentheses is therefore constant along the motion.

Noether's theorem does more than produce constants of motion: it identifies the conserved quantity from the sole data of the symmetry. This correspondence is functorial, in the sense that composing symmetries translates into adding the conserved quantities.

EXAMPLE : For a particle of mass m in a potential $V(|q|)$ in three dimensions: the translation $q \mapsto q + s\, e_j$ gives $\delta q^i = \delta^i_j$, and the conserved quantity is $p_j = m\dot{q}^j$ (momentum). The rotation $q \mapsto R_s q$ about the axis e_3 gives $\delta q = e_3 \times q$, and the conserved quantity is $L_3 = m(q^1 \dot{q}^2 - q^2 \dot{q}^1)$ (angular momentum). Time-translation invariance $t \mapsto t + s$ (when $\partial L / \partial t = 0$) produces conservation of the energy $E = \sum p_i \dot{q}^i - L$.

REMARQUE : The version presented here concerns point symmetries (acting on Q). Noether in fact established a more general result covering symmetries acting on curves in configuration space, including time reparametrisations. Her second theorem, distinct from the first, treats symmetries depending on arbitrary functions (gauge symmetries) and shows that they produce not conserved quantities but identities among the equations of motion. This distinction between global and local symmetries will structure the entire theory of gauge fields.

That energy conservation results from the homogeneity of time, and momentum conservation from the homogeneity of space: Noether's theorem transforms these intuitions into theorems and opens the way to reading physics as the geometry of symmetries.

004. LAGRANGIANS WITH CONSTRAINTS.

A pendulum swings: its mass moves in the plane, but the rigid rod constrains it to remain on a circle. Two coordinates describe the plane; one suffices for the circle. The constraint reduces the configuration space, and the Lagrangian formalism must accommodate this reduction. Two paths present themselves: eliminate the superfluous degrees of freedom by choosing adapted coordinates, or keep the original coordinates and add multipliers encoding the force exerted by the constraint. The first is the architecture of a building whose plan respects the terrain; the second, that of a free-standing building to which load-bearing walls are added.

DEFINITION 004.1: A *holonomic constraint* on a system of N point particles with positions $q_1, \dots, q_N \in \mathbb{R}^3$ is a relation of the form

$$f_\alpha(q_1, \dots, q_N, t) = 0, \qquad \alpha = 1, \dots, k,$$

where the f_α are smooth functions and the differentials df_α are linearly independent on the constraint surface. The constrained configuration space is a submanifold Q of dimension $n = 3N - k$.

Any local parametrisation $q = (q^1, \dots, q^n)$ of Q provides a system of *generalised coordinates*. In these coordinates, the constraints are automatically satisfied and the Lagrangian is written directly on TQ.

PROPOSITION (D'ALEMBERT'S PRINCIPLE): The constraint forces do no work in virtual displacements compatible with the constraints. More precisely, if δq is tangent to the constraint surface, then

$$\sum_{j=1}^{N} \left(m_j \ddot{q}_j - F_j\right) \cdot \delta q_j = 0,$$

where F_j denotes the applied (non-constraint) force on the j-th particle.

This principle, whose statement seems tautological for a rigid rod or a smooth surface, is in reality a physical hypothesis: it selects, among all possible forces perpendicular to the surface, those that are ideal (frictionless). It is what enables the passage from Newton to Lagrange.

THÉORÈME: Consider a system subject to holonomic constraints $f_\alpha(q,t) = 0$, $\alpha = 1, \ldots, k$. The equations of motion are obtained by rendering stationary the functional

$$S_\lambda[q,\lambda] = \int_{t_0}^{t_1} \left(L(q,\dot{q},t) + \sum_{\alpha=1}^{k} \lambda_\alpha(t) f_\alpha(q,t) \right) dt,$$

which gives the modified Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_{\alpha=1}^{k} \lambda_\alpha \frac{\partial f_\alpha}{\partial q^i}, \qquad f_\alpha(q,t) = 0.$$

The multipliers $\lambda_\alpha(t)$ are interpreted as the components of the constraint force.

The method of multipliers retains the information about constraint forces, which the reduction to generalised coordinates erases. When one needs to know the tension in a wire or the reaction of a surface, multipliers are indispensable.

EXAMPLE: *For the simple pendulum of length ℓ and mass m, the constraint is $x^2 + y^2 = \ell^2$. In generalised coordinates $q = \theta$ (angle from the vertical), the reduced Lagrangian is $L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$, and the equation of motion is $\ddot{\theta} + (g/\ell)\sin\theta = 0$. In Cartesian coordinates with a multiplier, one also obtains the tension of the rod $T = m(g\cos\theta + \ell\dot{\theta}^2)$.*

REMARQUE: Non-holonomic constraints (for example rolling without slipping, $\dot{x} = r\dot{\theta}$, which does not integrate to a relation $f(x,\theta) = 0$) cannot be treated by simple reduction of the configuration space. They require the Lagrange-d'Alembert equations, whose geometric structure involves non-integrable distributions on TQ. This distinction between integrable and non-integrable constraints is a mechanical avatar of the Frobenius theorem.
The theory of constraints prepares an idea that will run through the entire volume: reducing degrees of freedom via a symmetry or a constraint does not destroy the structure; it concentrates it on a smaller space, where the physics is more legible.

005. CALCULUS OF VARIATIONS: FOUNDATIONS.

IN ordinary analysis, one seeks the extrema of a function $f : \mathbb{R}^n \to \mathbb{R}$ by setting its gradient to zero. The calculus of variations poses the same problem in an infinite-dimensional space: the unknown is no longer a point but a curve, and the functional to be optimised is the integral of a Lagrangian along that curve. To give precise meaning to "differentiating with respect to a path," one must construct the infinite-dimensional analogue of the differential, which requires defining with care the space in which paths live and the notion of admissible variation. It is a passage from the finite alphabet of coordinates to the continuous grammar of functions.

DEFINITION 005.1: Let Q be a smooth manifold. The *path space* with fixed endpoints is

$$\mathcal{P}(q_0, q_1) = \{q \in C^2([t_0, t_1], Q) : q(t_0) = q_0,\ q(t_1) = q_1\}.$$

For $q \in \mathcal{P}(q_0, q_1)$ and $\eta \in C^2([t_0, t_1], T_q Q)$ with $\eta(t_0) = \eta(t_1) = 0$, the *Gateaux derivative* (or *first variation*) of the functional S is

$$\delta S[q] \cdot \eta = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S[q + \varepsilon\eta].$$

A path q^* is a *critical point* (or *extremal*) of S if $\delta S[q^*] \cdot \eta = 0$ for every variation η.

The boundary condition $\eta(t_0) = \eta(t_1) = 0$ is indispensable: without it, the integration by parts leading to the Euler-Lagrange equations would leave non-vanishing boundary terms.

THÉORÈME : Let $L(q, \dot{q}, t)$ be of class C^2. If q^* is an extremal of S in $\mathcal{P}(q_0, q_1)$, then q^* satisfies the Euler-Lagrange equations. Conversely, every solution of the Euler-Lagrange equations satisfying the boundary conditions is an extremal.

The vanishing of the first variation does not distinguish between a minimum, a maximum, and a saddle point. To classify extremals, one must examine the second variation.

PROPOSITION : The *second variation* of S at an extremal q^* is the bilinear form

$$\delta^2 S[q^*](\eta, \eta) = \int_{t_0}^{t_1} \left(P_{ij}\, \eta^i \eta^j + 2Q_{ij}\, \eta^i \dot{\eta}^j + R_{ij}\, \dot{\eta}^i \dot{\eta}^j \right) dt,$$

where $P_{ij} = \frac{\partial^2 L}{\partial q^i \partial q^j}$, $Q_{ij} = \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j}$, $R_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$, evaluated along q^*. The extremal is a local minimum (in the weak sense) if $\delta^2 S > 0$ for every variation $\eta \neq 0$. The *Legendre condition* $R_{ij}(t) > 0$ (positive definite matrix for all t) is a necessary condition.

The Legendre condition is necessary but not sufficient: the sufficient condition (Jacobi's condition) involves the *conjugate points*, instants at which a family of neighbouring solutions refocuses. Their absence on $[t_0, t_1]$ guarantees the positivity of $\delta^2 S$.

EXAMPLE: For the length functional $S[\gamma] = \int_0^1 |\dot\gamma(t)|\,dt$ *on a surface, the extremals are the geodesics. The second variation brings in the sectional curvature: on a sphere (positive curvature), geodesics refocus (conjugate points); on a hyperbolic plane (negative curvature), they never do.*

REMARQUE: A technical subtlety deserves attention. The extremal q^* is *a priori* of class C^2 by hypothesis, but the Du Bois-Reymond lemma shows that if L is C^k and the strong Legendre condition holds ($R_{ij} > 0$), then the extremal is in fact C^{k+1}. This elliptic regularity of extremals is the variational counterpart of the regularity of solutions to elliptic differential equations.

The calculus of variations thus provides the analytic foundation of all Lagrangian mechanics. Its concepts (path space, first and second variation, boundary conditions) will transpose, in infinite dimensions, to field theory, where the path is replaced by a field defined on spacetime.

006. FUNDAMENTAL EXAMPLES.

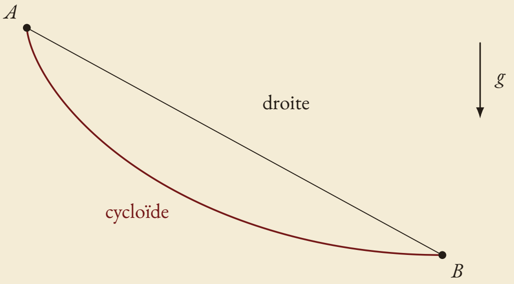
FOUR problems, posed between the seventeenth and eighteenth centuries, forged the calculus of variations before it had a name. The pendulum raises the question of constrained motion; the brachistochrone, that of the optimal path; geodesics, that of intrinsic distance; the catenary, that of equilibrium under gravity. Each illuminates a distinct aspect of Lagrangian theory, as four perspectives on a single edifice reveal in turn the facade, the structure, the plan, and the foundations.

EXAMPLE: The simple pendulum *of length* ℓ *and mass* m *is described by the angle* θ *from the vertical. The Lagrangian* $L = \frac{1}{2}m\ell^2\dot\theta^2 + mg\ell\cos\theta$ *gives the equation of motion* $\ddot\theta + (g/\ell)\sin\theta = 0$*. For small oscillations,* $\sin\theta \approx \theta$ *and the motion is harmonic with period* $2\pi\sqrt{\ell/g}$*. The exact solution involves Jacobi's elliptic functions, whose period depends on the amplitude: nonlinearity breaks isochronism.*

PROPOSITION (BRACHISTOCHRONE): Among all smooth curves connecting a point A to a point B located below it in a uniform gravitational field, the one along which a point mass sliding without friction descends most quickly is an arc of a *cycloid*. The Lagrangian of the problem is

$$T[y] = \int_0^{x_1} \sqrt{\frac{1 + y'(x)^2}{2g\,y(x)}}\,dx,$$

where y is the depth below A, and the optimal curve satisfies the associated Euler-Lagrange equation.



The brachistochrone: the cycloid descends faster than the straight line.

The brachistochrone, posed by Johann Bernoulli in 1696 as a challenge to the mathematicians of Europe, is the problem that gave birth to the calculus of variations. Its solution by the cycloid reveals that the fastest path is neither the straight line nor the shortest path: the curve first plunges steeply to gain speed, then rises toward the endpoint. The optimality of a path depends on the chosen criterion.

PROPOSITION (GEODESICS): On a regular surface $S \subset \mathbb{R}^3$ parametrised by (u^1, u^2) with induced metric g_{ij}, the curves of minimal length between two points satisfy the Euler-Lagrange equations of the energy functional

$$E[\gamma] = \frac{1}{2}\int_0^1 g_{ij}\,\dot{u}^i\,\dot{u}^j\,dt,$$

namely the *geodesic equations*:

$$\ddot{u}^k + \Gamma^k_{ij}\,\dot{u}^i\,\dot{u}^j = 0,$$

where Γ^k_{ij} are the Christoffel symbols of the metric.

Geodesics are to curved surfaces what straight lines are to the plane: the trajectories of free particles. Their study links the calculus of variations to differential geometry; the curvature of the surface is read from the behaviour of neighbouring geodesics, via the Jacobi equation.

EXAMPLE : *The* catenary *is the equilibrium curve of a flexible, homogeneous wire suspended between two points under gravity. It minimises the gravitational potential energy* $V[y] = \rho g \int_a^b y\sqrt{1+y'^2}\,dx$ *subject to the fixed-length constraint* $\int_a^b \sqrt{1+y'^2}\,dx = \ell$. *The Lagrange multiplier gives the equation* $y(x) = \alpha\cosh\big((x-x_0)/\alpha\big)$, *where* α *is determined by the length. Galileo believed it was a parabola; Leibniz, Huygens, and Johann Bernoulli independently showed in 1691 that it is a hyperbolic cosine.*

REMARQUE : These four problems illustrate the four types of variational functionals encountered in mechanics: the action functional (pendulum), the time functional (brachistochrone), the energy or length functional (geodesics), and a functional with an isoperimetric constraint (catenary). Each brings in a different aspect of the theory: classical Lagrangian, non-standard parameter, Riemannian geometry, Lagrange multipliers. The calculus of variations unifies them within a single framework, that of finding critical points of a functional on a space of curves.

From the swing of a pendulum to the shortest curves on a surface, the variational principle reveals itself to be not one method among many, but the natural language in which mechanics, geometry, and optimisation converge.

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This book was typeset in Garamond
in the pocket format 105 × 170 mm
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The symbol on the cover is the first letter of the Hebrew alphabet, the aleph. In mathematics, it denotes Cantor's transfinite cardinals. In literature, it is the point in Borges where the entire universe concentrates in a single place. Its name, in the Semitic languages, means "head of an ox." The reader who might see in that head the echo of a horned goddess carrying the sun between her two horizons would not be entirely wrong.