Articulated Course in Mathematics

IV

Correspondences

From Class Field Theory to the Langlands Programme

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Free sample — Chapter 1: Number Fields and Rings of Integers (6 articles).
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TO THE READER

This course articulates in five volumes and over twelve hundred articles the grammar by which reality composes itself, from the first whole number to the Langlands correspondence, from the brachistochrone to the amplituhedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.

Each article develops a single idea, from its motivation to its interpretation. Definitions are framed; theorems are also framed, in red. Proofs are present where they illuminate, sketched where full rigour would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The reader will have no trouble distinguishing the three cases.

Ten mathematical objects traverse the collection like red threads: the circle S^1*, the integers* $\mathbb{Z}$*, the extension* $\mathbb{Q}(\sqrt{2})$*, the elliptic curve* $y^2 = x^3 - x$*, the group* GL_2*, the function* $\zeta(s)$*, the space* L^2*, the ring* $K[X]$*, the symmetric group* $\mathfrak{S}_3$*, and the torus* T^2*. From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.*

The figures, numbering six hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colours. They do not replace demonstration; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.

The reader may follow the linear path or take the transversal passages between volumes. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands programme; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.

Every error reported is an error corrected.

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Table of Contents

Part I

Algebraic Arithmetic

CHAPTER I

NUMBER FIELDS AND RINGS OF INTEGERS

A number field is a finite extension of $\mathbb{Q}$; its ring of integers generalises $\mathbb{Z}$. Norm, trace, discriminant: these invariants measure the arithmetic complexity of the extension. Dedekind ideals restore unique factorisation by replacing it with a factorisation into prime ideals. This chapter lays the foundations of algebraic number theory, the terrain on which reciprocity laws and the Langlands programme are built.

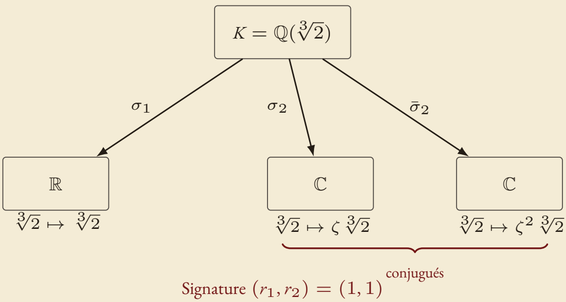
Articles in this chapter:

001. NUMBER FIELDS.

What happens when one adjoins $\sqrt{-5}$ to the rationals? One obtains a field $\mathbb{Q}(\sqrt{-5})$ in which the integer 6 admits two essentially distinct factorisations: $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. This arithmetic scandal (the collapse of unique factorisation) is the starting point of all algebraic number theory. To understand what is going on, one must first specify the playing field: number fields.

DEFINITION 001.1: A *number field* is a finite extension K of $\mathbb{Q}$. Its *degree* is the integer $n = [K : \mathbb{Q}]$. The field $\mathbb{Q}$ itself is a number field of degree 1. The quadratic fields $\mathbb{Q}(\sqrt{d})$ (with $d \in \mathbb{Z}$ squarefree) have degree 2, and the cyclotomic fields $\mathbb{Q}(\zeta_m)$ have degree $\varphi(m)$.

Every number field admits exactly n embeddings into $\mathbb{C}$, some real and others complex. Their distribution determines the arithmetic nature of the field.



The three embeddings of $\mathbb{Q}(\sqrt[3]{2})$*: one real and a pair of complex conjugates, giving signature* $(1,1)$.

DEFINITION 001.2: Let K be a number field of degree n. An *embedding* of K is a field homomorphism $\sigma : K \hookrightarrow \mathbb{C}$. There are exactly n of them. Among these, r_1 are *real* (with image contained in $\mathbb{R}$) and the remaining $n - r_1$ group into r_2 pairs of complex conjugates. The pair (r_1, r_2), with $r_1 + 2r_2 = n$, is the *signature* of K.

PROPOSITION: By the primitive element theorem, every number field can be written $K = \mathbb{Q}(\alpha)$ for some algebraic α, a root of an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree n. A basis of K over $\mathbb{Q}$ is then $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$.

The existence of a primitive element reduces the study of a number field to that of a single irreducible polynomial, but the proof requires separability of the extension: a fact that is automatic in characteristic zero.

PROOF: The extension $K/\mathbb{Q}$ is finite and separable (since $\mathbb{Q}$ has characteristic zero). The primitive element theorem applies directly. The irreducibility of f ensures $[K : \mathbb{Q}] = \deg f = n$, and the powers of α form a basis because $K \simeq \mathbb{Q}[x]/(f)$.

EXAMPLE: The field $\mathbb{Q}(\sqrt{2})$ *has signature* $(2, 0)$*: its two embeddings send* $\sqrt{2}$ *to* $\pm\sqrt{2}$*, both real. The field* $\mathbb{Q}(i)$ *has signature* $(0, 1)$*: its two embeddings send* i *to* $\pm i$ *and form a conjugate pair. The field* $\mathbb{Q}(\sqrt[3]{2})$ *has signature* $(1, 1)$*: one real cube root and two complex conjugate roots.*

REMARK: The signature (r_1, r_2) governs the rank of the unit group (Dirichlet's theorem: rank $r_1 + r_2 - 1$), the sign of the discriminant ($(-1)^{r_2}$) and the form of the canonical embedding of K into $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. It condenses in two integers the archimedean constraints that bear on all the arithmetic of K.

The field $\mathbb{Q}(\sqrt{2})$, which at the undergraduate level illustrated field extensions and at the master's level served as an elementary example in Galois theory, appears here as the simplest real number field (signature $(2, 0)$, ring of integers $\mathbb{Z}[\sqrt{2}]$, discriminant 8) and the testing ground on which every notion in this chapter (norm, trace, ideals, units) can be computed by hand.

The notion of a number field and its signature thus forms the bedrock on which all of algebraic number theory rests, from the ramification of primes to the deepest conjectures of the Langlands programme.

002. RINGS OF INTEGERS.

THE ordinary integers $\mathbb{Z}$ sit inside $\mathbb{Q}$ like the lattice points on a line. But when one passes to a number field K, which elements play the role of integers? The answer is not obvious: in $\mathbb{Q}(\sqrt{5})$, the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ (which hardly looks like an integer) satisfies $\varphi^2 - \varphi - 1 = 0$ and deserves the title of algebraic integer.

> DEFINITION 002.1: An *algebraic integer* is a complex number α that is a root of a monic polynomial with coefficients in $\mathbb{Z}$: $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$ with $a_i \in \mathbb{Z}$. The set of all algebraic integers forms a subring $\overline{\mathbb{Z}}$ of $\mathbb{C}$.

The modular characterisation clarifies this notion: α is an algebraic integer if and only if $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module. Integrality is read off from the finiteness of the generated module.

> DEFINITION 002.2: Let K be a number field. The *ring of integers* of K is $\mathcal{O}_K = K \cap \overline{\mathbb{Z}}$, the set of elements of K that are algebraic integers.

PROPOSITION: If K is a number field of degree n, then $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n. In other words, there exists an *integral basis* $\{\omega_1, \ldots, \omega_n\}$ such that $\mathcal{O}_K = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$.
The free module structure is the fundamental property that allows one to speak of integral bases and discriminants. Its proof combines the arithmetic of algebraic integers with the properties of finitely generated modules over $\mathbb{Z}$.

PROOF: Any $\mathbb{Q}$-basis of K consisting of integers generates a submodule $M \subset \mathcal{O}_K$ of rank n. By a discriminant argument, $\mathcal{O}_K$ is contained in $\frac{1}{d}M$ for some nonzero integer d. Since $\mathbb{Z}$ is a principal ideal domain, every submodule of a free module of finite rank is free of rank at most n. Finally, $\mathcal{O}_K$ contains M, so its rank is exactly n.

EXAMPLE: *For $K = \mathbb{Q}(\sqrt{d})$ with d squarefree, the ring of integers is $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$ if $d \equiv 2$ or $3 \pmod 4$, and $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ if $d \equiv 1 \pmod 4$. The Gaussian integers $\mathbb{Z}[i]$ (case $d = -1$) are the most classical example. For cyclotomic fields, $\mathcal{O}_{\mathbb{Q}(\zeta_p)} = \mathbb{Z}[\zeta_p]$ when p is prime.*

REMARK: Determining $\mathcal{O}_K$ is a nontrivial algorithmic problem, even for cubic fields. The inclusion $\mathbb{Z}[\alpha] \subset \mathcal{O}_K$ can be strict: the ring of integers is larger than a naive generator suggests, and it is in this surplus that the fine arithmetic structure hides.
The ring of integers is the first genuinely arithmetic object one encounters: its structure as a free $\mathbb{Z}$-module of finite rank is the starting point for the entire theory of ideals, ramification, and ultimately Galois representations.

003. NORM, TRACE, DISCRIMINANT.

How does one measure the size of an algebraic element? In $\mathbb{Q}$, the absolute value suffices. But in a number field K of degree n, a single number lives simultaneously in n copies of $\mathbb{C}$ via the embeddings $\sigma_1, \ldots, \sigma_n$. The trace adds these incarnations, the norm multiplies them, and the discriminant measures how sharply the embeddings distinguish one another.

DEFINITION 003.1: Let K be a number field of degree n, with embeddings $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{C}$. For $\alpha \in K$, the *trace* and the *norm* are

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha), \qquad \mathrm{N}_{K/\mathbb{Q}}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha).$$

The trace is $\mathbb{Q}$-linear, the norm is multiplicative, and both send $\mathcal{O}_K$ into $\mathbb{Z}$.

The trace is the additive measure of an element, the norm its multiplicative measure. Together they determine the minimal polynomial in degree 2, and in arbitrary degree they fix its first and last coefficients.

DEFINITION 003.2: Let $\{\omega_1, \dots, \omega_n\}$ be an integral basis of $\mathcal{O}_K$. The *discriminant* of K is the integer

$$d_K = \det\big(\mathrm{Tr}_{K/\mathbb{Q}}(\omega_i\omega_j)\big)_{1\le i,j\le n} = \det\big(\sigma_i(\omega_j)\big)^2.$$

It is nonzero and independent of the choice of integral basis.

PROPOSITION: A prime p is ramified in K if and only if $p \mid d_K$. In particular, only finitely many primes ramify in a given extension. The link between ramification and the discriminant is one of the most useful results in the theory: it allows one to locate ramification from a single global invariant.

PROOF: The prime p ramifies if and only if $p\mathcal{O}_K$ contains the square of a prime ideal, which is equivalent to the vanishing of the discriminant of $\mathcal{O}_K/p\mathcal{O}_K$ over $\mathbb{F}_p$, itself equivalent to $p \mid d_K$. Finiteness follows from the fact that d_K has only finitely many prime divisors.

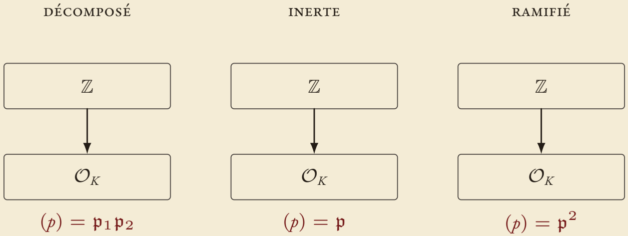
EXAMPLE: In $\mathbb{Q}(\sqrt{d})$: $d_K = d$ if $d \equiv 1 \pmod 4$ and $d_K = 4d$ otherwise. Thus $d_{\mathbb{Q}(i)} = -4$ (only 2 ramifies) and $d_{\mathbb{Q}(\sqrt{5})} = 5$ (only 5 ramifies). For $\mathbb{Q}(\zeta_p)$, one has $d_K = \pm p^{p-2}$: the only ramified prime is p itself, but with massive ramification.

REMARK: The discriminant also encodes the signature: its sign is $(-1)^{r_2}$. More deeply, it appears in the denominator of the analytic class number formula and in the volume of the adelic quotient $\mathbb{A}_K/K$. A single integer thus captures the geometric, arithmetic and analytic complexity of the field.

The discriminant, the trace and the norm form the triad of numerical invariants that accompanies every algebraic extension and whose avatars reappear in the conductor formula, the epsilon factors and the theory of automorphic representations.

004. DEDEKIND IDEALS.

In $\mathbb{Z}[\sqrt{-5}]$, the integer 6 factors in two irreducible ways: $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. A norm argument shows that 2, 3, $1 \pm \sqrt{-5}$ are irreducible, and the two factorisations are essentially distinct. This is a catastrophe: unique factorisation, the cornerstone of the arithmetic of $\mathbb{Z}$, collapses. Dedekind's idea, as simple as it is revolutionary, is to replace elements by ideals.



Factorisation of a prime: split, inert or ramified

DEFINITION 004.1: An integral domain A is a *Dedekind domain* if it is (i) Noetherian, (ii) integrally closed in its fraction field, and (iii) of Krull dimension at most 1: every nonzero prime ideal is maximal.

PROPOSITION: For every number field K, the ring $\mathcal{O}_K$ is a Dedekind domain. Noetherianity follows from the fact that $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$-module; integral closure results from the transitivity of algebraic integers; dimension 1 holds because $\mathcal{O}_K/\mathfrak{p}$ is a finite field for every nonzero prime $\mathfrak{p}$.

The fundamental theorem restores uniqueness, not for elements, but for ideals.

proposition: In a Dedekind domain, every nonzero ideal $\mathfrak{a}$ decomposes uniquely as a product of prime ideals:

$$\mathfrak{a} = \mathfrak{p}_1^{e1}\mathfrak{p}_2^{e2}\cdots\mathfrak{p}_r^{er}, \qquad e_i \geq 1.$$

More generally, the nonzero fractional ideals form a free abelian group with basis the nonzero prime ideals.
The existence and uniqueness of this factorisation constitute the central result of the theory of Dedekind domains. The existence proof uses Noetherianity; uniqueness uses integral closure.

proof: Existence proceeds by contradiction and Noetherianity: a maximal element among ideals that do not factor leads to a contradiction. For uniqueness, one shows that each prime ideal $\mathfrak{p}$ admits a fractional inverse $\mathfrak{p}^{-1}$ with $\mathfrak{p}\mathfrak{p}^{-1} = \mathcal{O}_K$: this is where integral closure intervenes. One then argues by induction, multiplying by $\mathfrak{p}_1^{-1}$.

example: Return to $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. The prime ideals $\mathfrak{p}_2 = (2, 1 + \sqrt{-5})$, $\mathfrak{p}_3 = (3, 1 + \sqrt{-5})$ and $\bar{\mathfrak{p}}_3 = (3, 1 - \sqrt{-5})$ satisfy $(2) = \mathfrak{p}_2^2$, $(3) = \mathfrak{p}_3\bar{\mathfrak{p}}_3$, $(1 + \sqrt{-5}) = \mathfrak{p}_2\mathfrak{p}_3$ and $(1 - \sqrt{-5}) = \mathfrak{p}_2\bar{\mathfrak{p}}_3$. The two factorisations of 6 yield the same product of ideals: $(6) = \mathfrak{p}_2^2\mathfrak{p}_3\bar{\mathfrak{p}}_3$.

remark: Uniqueness at the level of ideals rests on the existence of the fractional inverse $\mathfrak{p}^{-1}$, which in turn uses the integral closure of $\mathcal{O}_K$. If one weakens this hypothesis (for instance in an order $\mathbb{Z}[\sqrt{-3}] \subsetneq \mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$), the factorisation into prime ideals fails. The Dedekind condition is exactly the boundary of uniqueness.
The ring $\mathbb{Z}$, prototype of all Dedekind domains encountered as early as secondary school as the arena of arithmetic, yielded unique factorisation of elements by virtue of being a principal ideal domain. Here, in $\mathcal{O}_K$ for a general number field, it is the factorisation of *ideals* that restores uniqueness, and the class group measures exactly the distance between $\mathcal{O}_K$ and the lost paradise of $\mathbb{Z}$.

Unique factorisation of ideals in Dedekind domains is the foundation on which class field theory, the decomposition of primes in extensions, and further afield, the very definition of Artin *L*-functions all rest.

005. CLASS GROUP.

If every ideal of $\mathcal{O}_K$ were principal, unique factorisation of elements would be automatic. The obstacle is that some ideals are not: in $\mathbb{Z}[\sqrt{-5}]$, the ideal $(2, 1+\sqrt{-5})$ is generated by no single element. The class group quantifies this gap exactly: its order h_K measures how far the arithmetic of $\mathcal{O}_K$ deviates from that of $\mathbb{Z}$. This number, always finite by a geometry-of-numbers argument, is one of the most important invariants of a number field. It links the ideal structure of $\mathcal{O}_K$ to the analytic class number formula and, through it, to the residue of the Dedekind zeta function.

DEFINITION 005.1: The *class group* of the number field K is the quotient

$$\mathrm{Cl}(K) = \frac{\{\text{nonzero fractional ideals of } \mathcal{O}_K\}}{\{\text{principal fractional ideals}\}}.$$

It is a finite abelian group. Its order $h_K = |\mathrm{Cl}(K)|$ is the *class number*. One has $h_K = 1$ if and only if $\mathcal{O}_K$ is a principal ideal domain, that is, if the classical unique factorisation works.

PROPOSITION: Every ideal class contains an ideal of norm at most

$$C_K = \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r2}\sqrt{|d_K|},$$

the *Minkowski bound*. Since there are only finitely many ideals of bounded norm, the class number is finite.

The Minkowski bound, drawn from the geometry of numbers, is the concrete tool that makes the class group computable: one need only test finitely many ideals.

PROOF: Let $\mathfrak{a}$ be a nonzero ideal. Minkowski's lemma, applied to the lattice $\mathfrak{a}$ in $K_\infty \simeq \mathbb{R}^n$, produces a nonzero $\alpha \in \mathfrak{a}$ with $|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| \leq C_K \cdot \mathrm{N}(\mathfrak{a})$. The ideal $\mathfrak{b} = \alpha\mathfrak{a}^{-1}$ lies in the inverse class of $\mathfrak{a}$ and satisfies $\mathrm{N}(\mathfrak{b}) \leq C_K$. Since the ideals of norm $\leq C_K$ are finite in number, the result follows.

EXAMPLE: The field $\mathbb{Q}(i)$ has $h = 1$: the ring $\mathbb{Z}[i]$ is even Euclidean. The field $\mathbb{Q}(\sqrt{-5})$ has $h = 2$, with $\mathrm{Cl}(K) \simeq \mathbb{Z}/2\mathbb{Z}$ generated by the class of $(2, 1+\sqrt{-5})$. The field $\mathbb{Q}(\sqrt{-163})$ has $h = 1$, a remarkable fact related to the Heegner–Stark theorem. There are exactly nine imaginary quadratic fields of class number 1.

REMARK: The analytic class number formula relates h_K to the residue at $s = 1$ of the Dedekind zeta function: $\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|d_K|}}$. This bridge between an algebraic object and an analytic one is the ancestor of all the correspondences in the Langlands programme.

The class group, which measures the deviation between the arithmetic of a number field and that of $\mathbb{Z}$, reappears in various guises throughout the Langlands programme: in class field theory, in Selmer groups, and in the Birch and Swinnerton-Dyer conjecture.

006. DIRICHLET UNITS.

THE equation $x^2 - 2y^2 = 1$ (the Pell equation) admits infinitely many integer solutions, generated by the fundamental solution $(3, 2)$. This elementary fact conceals a deep theorem: the units of $\mathbb{Z}[\sqrt{2}]$ form an infinite group, generated (up to sign) by $1 + \sqrt{2}$. Dirichlet's theorem generalises this observation to every number field, revealing the exact structure of the unit group.

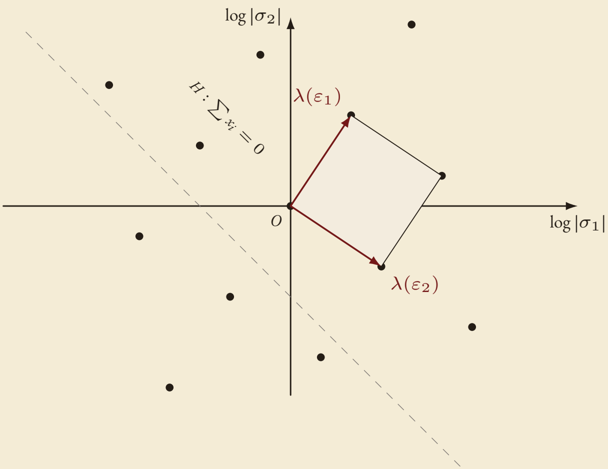
DEFINITION 006.1: The *units* of $\mathcal{O}_K$ are the invertible elements: $\mathcal{O}_K^\times = \{\alpha \in \mathcal{O}_K : \mathrm{N}_{K/\mathbb{Q}}(\alpha) = \pm 1\}$.

PROPOSITION: Let K be a number field of signature (r_1, r_2). The unit group is a finitely generated abelian group:

$$\mathcal{O}_K^\times \simeq \mu_K \times \mathbb{Z}^{r_1+r_2-1},$$

where μ_K is the finite cyclic group of roots of unity in K.
The key to the proof is the logarithmic embedding, which transforms the multiplicative problem into a lattice problem in Euclidean space.

PROOF: The logarithmic map $\lambda : \mathcal{O}_K^\times \to \mathbb{R}^{r_1+r_2}$ defined by $\alpha \mapsto (\log|\sigma_i(\alpha)|)$ sends the units into the hyperplane H with equation $\sum x_i = 0$ (since $|\mathrm{N}(\alpha)| = 1$). The kernel of λ is μ_K: the units of absolute value 1 under every embedding. A geometry-of-numbers argument shows that the image is a lattice in H, hence a free group of rank $\dim H = r_1 + r_2 - 1$.
The logarithmic image of the units forms a lattice in a hyperplane: it is this discrete geometric structure that makes the unit group computable.



Lattice of units in the hyperplane $H : \sum x_i = 0$, with vectors $\lambda(\varepsilon_1)$ and $\lambda(\varepsilon_2)$ generating the lattice.

The figure above shows the lattice generated by the logarithmic images of the fundamental units: the regulator measures the area of the fundamental parallelogram.

DEFINITION 006.3: If $\varepsilon_1, \ldots, \varepsilon_r$ (with $r = r_1 + r_2 - 1$) are fundamental units, the *regulator* of K is $R_K = |\det(\lambda_i(\varepsilon_j))_{1 \leq i,j \leq r}|$, obtained by omitting one coordinate. It measures the covolume of the unit lattice in the hyperplane H.

EXAMPLE: For $\mathbb{Q}(\sqrt{2})$*:* $r_1 = 2$*,* $r_2 = 0$*,* $r = 1$*. The fundamental unit is* $\varepsilon = 1 + \sqrt{2}$ *(of norm* -1*) and* $R_K = \log(1 + \sqrt{2}) \approx 0.88$*. For* $\mathbb{Q}(i)$*:* $r = 0$*, there is no free part, and* $\mathcal{O}_K^\times = \{1, i, -1, -i\}$*. For* $\mathbb{Q}(\sqrt[3]{2})$*:* $r = 1$ *and the fundamental unit is* $1 - \sqrt[3]{2}$*.*

REMARK: The regulator appears in the numerator of the class number formula, in tandem with h_K. A large regulator compensates for a small class number, and conversely. This arithmetic balance between the size of the unit group and the failure of principality is one of the most subtle phenomena in number theory.
The logarithmic lattice of units, whose covolume is the regulator, is one of the finest invariants of the arithmetic of a number field. Its contribution to the analytic class number formula illustrates the permanent dialogue between algebraic structure and analytic information that runs through the entire theory.

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The symbol on the cover is the first letter of the Hebrew alphabet, the aleph. In mathematics, it denotes Cantor's transfinite cardinals. In literature, it is the point in Borges where the entire universe concentrates in a single place. Its name, in the Semitic languages, means "head of an ox." The reader who might see in that head the echo of a horned goddess carrying the sun between her two horizons would not be entirely wrong.