# Articulated Course

# in

# Mathematics

III

# Maîtrise

From Structure to Depth

Clementinium Editions

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*Free sample — Chapter 1: Topology of Function Spaces (5 articles).*
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# *TO THE READER*

*This course articulates in five volumes and over twelve hundred articles the grammar by which reality composes itself, from the first whole number to the Langlands correspondence, from the brachistochrone to the amplituhedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.*

*Each article develops a single idea, from its motivation to its interpretation. Definitions are framed; theorems are also framed, in red. Proofs are present where they illuminate, sketched where full rigour would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The reader will have no trouble distinguishing the three cases.*

*Ten mathematical objects traverse the collection like red threads: the circle* $S^1$*, the integers* $\mathbb{Z}$*, the extension* $\mathbb{Q}(\sqrt{2})$*, the elliptic curve* $y^2 = x^3 - x$*, the group* $\mathrm{GL}_2$*, the function* $\zeta(s)$*, the space* $L^2$*, the ring* $K[X]$*, the symmetric group* $\mathfrak{S}_3$*, and the torus* $T^2$*. From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.*

*The figures, numbering six hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colours. They do not replace demonstration; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.*

*The reader may follow the linear path or take the transversal passages between volumes. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands programme; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.*

*Every error reported is an error corrected.*

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# Table of Contents

## Convexity and Optimisation

## Fixed Points and Applications

## Measure and Integration

## Probability and Stochastic Processes

## Mathematical Statistics

## Differential Equations

## Dynamical Systems

## Distributions and Fourier Analysis

## Advanced Complex Analysis

## Advanced Linear Algebra

## Bilinear Geometry

## General Algebra

## Galois Theory

## Structures and Commutative Algebra

## Topology and Differential Geometry

## Algebraic Geometry

## Numerical Analysis

## Scientific Computing and Modelling

## Reading and Working

Transition Exercises

# Part I

# Functional Analysis

# *CHAPTER* I

# *TOPOLOGY OF FUNCTION SPACES*

*Function spaces are not like other spaces: they are often infinite-dimensional, and their topology holds surprises. The Baire theorem, Ascoli's theorem, Stone-Weierstrass: these classical results characterise completeness, compactness, and density in function spaces. This chapter lays the topological foundations of functional analysis and shows that the properties of function spaces govern the very possibility of solving the equations that live within them.*

*Articles in this chapter :*

## *001.* *THE BAIRE CATEGORY THEOREM AND APPLICATIONS.*

*Can* one exhaust a complete metric space by covering it, piece by piece, with negligible closed sets? Intuition rebels against the idea, and the Baire category theorem vindicates that intuition. A complete space possesses a topological solidity that no countable accumulation of dust can erode. This result, modest in appearance, turns out to be the foundation on which the major theorems of functional analysis rest.

**DEFINITION 001.1:** Let $(X, d)$ be a metric space. A subset $A \subset X$ is said to have *empty interior* (or to be *nowhere dense*) if $\overline{A}$ has empty interior, that is $\overset{\circ}{\overline{A}} = \varnothing$. A set is *meagre* (or *of first category*) if it is a countable union of nowhere dense sets. A set is *residual* if its complement is meagre.

With this definition in place, the central result reads as follows.

**THEOREM (BAIRE):** Let $(X, d)$ be a complete metric space. Then:

1. every countable intersection of dense open sets is dense in $X$;
2. $X$ is not meagre in itself.

**PROOF:** Let $(U_n)_{n\geq 1}$ be a sequence of dense open subsets of $X$ and $B(x_0, r_0)$ a nonempty open ball. Since $U_1$ is dense, there exist $x_1 \in U_1 \cap B(x_0, r_0)$ and $r_1 < r_0/2$ such that $\overline{B(x_1, r_1)} \subset U_1 \cap B(x_0, r_0)$. By induction, one constructs a sequence of nested closed balls $\overline{B(x_n, r_n)}$ with $r_n < r_0/2^n$ and $\overline{B(x_n, r_n)} \subset U_n$. The sequence $(x_n)$ is Cauchy; by completeness, it converges to a point $x \in \bigcap_{n\geq 1} U_n \cap B(x_0, r_0)$. The second assertion follows: if $X = \bigcup F_n$ with each $F_n$ closed and of empty interior, the open sets $U_n = X \setminus F_n$ would be dense and their intersection would be dense, hence nonempty, contradicting $\bigcap U_n = \varnothing$.

The following consequence is immediate.

**PROPOSITION:** Let $(X, d)$ be a complete metric space. If $X = \bigcup_{n\geq 1} F_n$ with each $F_n$ closed, then at least one $F_n$ has nonempty interior.

This direct consequence of the Baire theorem is the lever that shifts many arguments from pointwise to uniform. It underpins the Banach-Steinhaus theorem: if a family of continuous linear maps between Banach spaces is pointwise bounded, the set where the norm stays controlled cannot be meagre, and the bound becomes uniform.

EXAMPLE: *The set of continuous functions on* $[0, 1]$ *that are differentiable at no point forms a residual subset of* $C([0, 1])$. *The existence of such functions, originally demonstrated by Weierstrass, thus follows from a category argument: the set of functions differentiable at least at one point is meagre.*

REMARK: The Baire theorem extends to locally compact spaces (not necessarily metrisable). The notion of « Baire category » classifies subsets of a topological space into meagre and non-meagre, a topological analogue of the dichotomy between null sets and sets of positive measure. A space is called a *Baire space* if every nonempty open subset is non-meagre; complete metric spaces and locally compact spaces are Baire. It is this structural property that, like an underground current, feeds the major results one will encounter in chapter 3.

One should remember that completeness is the lever transforming countable arguments into uniform results. A principle whose reach will become apparent throughout this volume.

### 002. THE ASCOLI THEOREM.

*How* does one recognise, within the immensity of $C(K)$, those families of functions from which a convergent subsequence can be extracted? In finite dimension, the answer is transparent: closed and bounded suffices. In infinite dimension, this characterisation collapses. Boundedness no longer compensates for the failure of compactness of the unit ball. An additional ingredient is needed, one that controls not the size of the functions but the coherence of their oscillations: equicontinuity.

DEFINITION 002.1: Let $K$ be a compact metric space and $(E, d)$ a metric space. A family $\mathcal{F} \subset C(K, E)$ is called *equicontinuous* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $f \in \mathcal{F}$ and all $x, y \in K$ satisfying $d_K(x, y) < \delta$, one has $d(f(x), f(y)) <$

$\varepsilon$. The family $\mathcal{F}$ is called *pointwise bounded* if, for every $x \in K$, the set $\{f(x) : f \in \mathcal{F}\}$ is bounded in $E$.

With this definition in place, the central result reads as follows.

*THEOREM (ASCOLI-ARZELA):* Let $K$ be a compact metric space and $\mathcal{F} \subset C(K, \mathbb{R})$ equipped with the uniform norm. Then $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded.

*PROOF:* *Necessary condition.* If $\overline{\mathcal{F}}$ is compact, it is totally bounded. For every $\varepsilon > 0$, one covers $\overline{\mathcal{F}}$ by finitely many balls of radius $\varepsilon/3$ centred at $f_1, \ldots, f_N$. Each $f_i$ is uniformly continuous on the compact set $K$, so there exists $\delta_i > 0$ such that $d_K(x, y) < \delta_i$ implies $|f_i(x) - f_i(y)| < \varepsilon/3$. Setting $\delta = \min_i \delta_i$ yields the equicontinuity of $\mathcal{F}$. The pointwise bound follows from total boundedness.
*Sufficient condition.* Let $(f_n)$ be a sequence in $\mathcal{F}$. Fix $D = \{x_k\}$ a countable dense subset of $K$ (which exists by compactness). By the pointwise bound and a diagonal extraction, one obtains a subsequence $(f_{n_j})$ converging pointwise on $D$. Equicontinuity then promotes this pointwise convergence to uniform convergence on all of $K$: for $\varepsilon > 0$, one chooses $\delta$ from equicontinuity, covers $K$ by finitely many balls $B(x_{k_i}, \delta)$, and convergence on the $x_{k_i}$ controls convergence everywhere.
Equicontinuity thus acts as a bridge between two worlds: it allows one to cross the gap separating pointwise convergence (easily obtained by diagonal extraction) from uniform convergence, the only convergence that has meaning in $C(K)$. The full force of the theorem lies in this promotion.

*EXAMPLE:* *Let $(f_n)$ be a sequence of K-Lipschitz functions on $[0, 1]$ with $|f_n(0)| \leq M$. The inequality $|f_n(x)| \leq M + K$ gives the pointwise bound, and $|f_n(x) - f_n(y)| \leq K|x - y|$ gives equicontinuity. By Ascoli, one can extract a uniformly convergent subsequence. This scheme lies at the heart of the proof of Peano's theorem for ordinary differential equations.*

*REMARK:* The Ascoli theorem is only the first chapter of a larger story. In finite dimension, Bolzano-Weierstrass suffices; in $C(K)$, Ascoli takes over; in $L^p$, Riesz-Frechet-Kolmogorov; in Sobolev spaces, Rellich-Kondrachov. Each function space demands its own compactness criterion, but the deep structure remains the same: a quantitative bound paired with a uniform regularity condition. The Ascoli theorem gives the mathematician the means to force compactness where infinite dimension seemed to forbid it: one need only control the oscillation.

### *003. THE STONE-WEIERSTRASS THEOREM.*

*WHY* can polynomials uniformly approximate every continuous function on a closed interval? The classical Weierstrass theorem asserts this, but its deep reason remains opaque as long as one views polynomials as explicit objects. Stone uncovered the secret: density depends not on the form of the polynomials, but on the algebraic properties of the family they generate: separating points and containing constants. This observation elevates a concrete result into an abstract principle of considerable reach.

*DEFINITION 003.1:* Let $K$ be a compact space and $\mathcal{A} \subset C(K, \mathbb{R})$. One says that $\mathcal{A}$ is a *subalgebra* of $C(K, \mathbb{R})$ if $\mathcal{A}$ is a vector subspace closed under multiplication. One says that $\mathcal{A}$ *separates points* of $K$ if, for all $x \neq y$ in $K$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

With this definition in place, the central result reads as follows.

*THEOREM (STONE-WEIERSTRASS, REAL VERSION):* Let $K$ be a compact space and $\mathcal{A}$ a subalgebra of $C(K, \mathbb{R})$ that contains the constant functions and separates points of $K$. Then $\mathcal{A}$ is dense in $C(K, \mathbb{R})$ under the uniform norm.

*PROOF:* The central idea is to show that the closure $\overline{\mathcal{A}}$ is a lattice. One proceeds in three steps. (1) One approximates $|t|$ uniformly on $[-1, 1]$ by polynomials $p_n(t)$ (classical Weierstrass or direct construction). (2) If $f \in \overline{\mathcal{A}}$, then $|f| = \lim p_n \circ (f/\|f\|_\infty) \cdot \|f\|_\infty \in \overline{\mathcal{A}}$, since

$\overline{\mathcal{A}}$ is a closed algebra. It follows that $\max(f,g) = \frac{f+g+|f-g|}{2}$ and $\min(f,g) = \frac{f+g-|f-g|}{2}$ belong to $\overline{\mathcal{A}}$. (3) Since $\mathcal{A}$ separates points and contains constants, for all $x \neq y$ and all reals $\alpha, \beta$, there exists $h \in \mathcal{A}$ with $h(x) = \alpha$ and $h(y) = \beta$. A lattice argument (localisation lemma) then allows one to approximate any continuous function.
This result extends to the following statement.

*THEOREM (STONE-WEIERSTRASS, COMPLEX VERSION):* Let $K$ be a compact space and $\mathcal{A}$ a subalgebra of $C(K, \mathbb{C})$ that contains the constants, separates points, and is *self-adjoint* (closed under complex conjugation: $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$). Then $\mathcal{A}$ is dense in $C(K, \mathbb{C})$.
*This result is admitted.*
The self-adjointness hypothesis is indispensable in the complex version: without it, the theorem fails. Complex analysis imposes its own rigidity here.

*EXAMPLE: (1) The algebra of polynomials* $\mathbb{R}[x]$ *separates points of any interval* $[a,b]$ *and contains the constants: one recovers the Weierstrass theorem. (2) The algebra of trigonometric polynomials* $\{x \mapsto \sum_{k=-N}^{N} c_k e^{ikx}\}$ *is a self-adjoint subalgebra of* $C(\mathbb{T}, \mathbb{C})$ *that separates points of the circle; it is therefore dense, which is the foundation of Fourier analysis. (3) The algebra* $\{f \in C(\overline{\mathbb{D}}, \mathbb{C}) : f \text{ holomorphic on } \mathbb{D}\}$ *separates points but is not self-adjoint; it is* not *dense in* $C(\overline{\mathbb{D}}, \mathbb{C})$.

*REMARK:* Each of the three hypotheses (point separation, containing constants, self-adjointness in the complex case) is necessary, and each guards against a distinct mode of failure. The disc algebra shows that closure under conjugation is not superfluous; omitting constants prevents free prescription of values; failing to separate points confuses distinct evaluations. The Stone-Weierstrass theorem thus teaches that density is an algebraic phenomenon before it is an analytic one.
The lesson of Stone-Weierstrass is that an algebra separating points contains, in germ, all continuous functions. Density is a matter of structure, not of construction.

### *004. COMPACTNESS IN FUNCTION SPACES.*

THE closed unit ball of a normed vector space is compact if and only if the space is finite-dimensional. This fact, proved by Riesz, marks a sharp break with intuition forged in finite dimension: in infinite dimension, being closed and bounded no longer guarantees compactness. Each function space must therefore find its own answer to the question: what must one add to boundedness to recover compactness? Like a key that changes shape according to the lock, the additional condition is always a uniform regularity, but its precise nature varies from one space to another.

*PROPOSITION (RIESZ):* Let $E$ be a normed vector space. The closed unit ball $\overline{B}(0,1)$ is compact if and only if $E$ is finite-dimensional.

*PROOF:* If $E$ is infinite-dimensional, Riesz's lemma provides, for every $\varepsilon \in (0,1)$, a sequence $(x_n)$ in the unit ball with $\|x_n - x_m\| \geq 1 - \varepsilon$ for $n \neq m$. No subsequence can be Cauchy, so $\overline{B}(0,1)$ is not sequentially compact.

In $C(K)$, the Ascoli theorem provides the answer: equicontinuity and pointwise boundedness characterise relatively compact subsets. In $L^p$, the corresponding criterion takes a different form, but the structure remains the same.

*THEOREM (RIESZ-FRECHET-KOLMOGOROV):* Let $1 \leq p < \infty$ and $\mathcal{F} \subset L^p(\mathbb{R}^n)$. Then $\mathcal{F}$ is relatively compact in $L^p(\mathbb{R}^n)$ if and only if:

1. $\mathcal{F}$ is bounded in $L^p$;
2. $\sup_{f\in\mathcal{F}} \|f(\cdot + h) - f(\cdot)\|_{L^p} \to 0$ as $h \to 0$ (*equicontinuity in translation*);
3. $\sup_{f\in\mathcal{F}} \int_{|x|>R} |f(x)|^p \, dx \to 0$ as $R \to \infty$ (*tightness*).

*This result is admitted.*

Equicontinuity in translation plays in $L^p$ the exact role of pointwise equicontinuity in $C(K)$: it prevents oscillations from refining without control. Tightness, in turn, prevents mass from escaping to infinity. This phenomenon is specific to unbounded domains, absent from the compact setting of Ascoli.

*EXAMPLE: The Sobolev embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact when $\Omega \subset \mathbb{R}^n$ is a bounded regular open set (Rellich-Kondrachov theorem). The bound in $H^1$ controls both the $L^2$ norm and the regularity (via the gradient), ensuring equicontinuity in translation. Every bounded sequence in $H^1(\Omega)$ therefore admits a subsequence converging in $L^2(\Omega)$.*

*REMARK:* The three criteria (Ascoli in $C(K)$, Riesz-Frechet-Kolmogorov in $L^p$, Rellich-Kondrachov in Sobolev spaces) share a common architecture: compactness arises from the conjunction of a quantitative bound and a uniform regularity condition. In nonlinear analysis and the calculus of variations, these results are the principal tool for turning minimising sequences into solutions, passing from bounded to convergent by way of compactness.

These compactness criteria form the indispensable toolkit for anyone seeking to extract a convergent limit from a bounded sequence in infinite dimension.

### 005. COMPLETE AND POLISH SPACES.

The open interval $(0,1)$, equipped with its usual distance, is not complete: the sequence $(1/n)$ is Cauchy but does not converge. Yet $(0,1)$ is homeomorphic to the whole of $\mathbb{R}$, which is complete. Completeness therefore depends on the chosen metric, not on the topology alone. Polish spaces arise from this observation: they are separable spaces that *admit* a complete compatible metric, without requiring every metric to be complete. This flexibility makes them the natural setting for measure theory and descriptive set theory.

*DEFINITION 005.1:* A topological space $X$ is called *Polish* if it is separable (it admits a countable dense subset) and *completely metrisable* (there exists a distance $d$ on $X$ compatible with the topology for which $(X,d)$ is complete).

The distinction between complete metrisability and completeness is essential. For $(0,1)$, the distance $d(x,y) = |\tan(\pi x - \pi/2) -$

$\tan(\pi y - \pi/2)|$ makes it complete; the space is therefore Polish. The topology is the same; only the metric changes.

*THEOREM (COMPLETION):* Every metric space $(X, d)$ admits a completion $(\hat{X}, \hat{d})$: a complete metric space into which $X$ embeds isometrically as a dense subset. This completion is unique up to isometry.

*PROOF:* One constructs $\hat{X}$ as the set of equivalence classes of Cauchy sequences in $X$, where $(x_n) \sim (y_n)$ if $d(x_n, y_n) \to 0$. The distance $\hat{d}([(x_n)], [(y_n)]) = \lim d(x_n, y_n)$ is well defined by the Cauchy property. The map $x \mapsto [(x, x, x, \ldots)]$ realises the isometric embedding, and its image is dense by construction. Uniqueness follows from the extension of isometries between dense subsets.

The following result is a consequence.

*PROPOSITION (ALEXANDROV):* Every open subset of a Polish space is Polish. More generally, every $G_\delta$ subset (countable intersection of open sets) of a Polish space is Polish.

*This result is admitted.*

This result is remarkable: an open subset of a complete space is in general not complete for the ambient metric, but it admits a compatible metric that makes it complete. The trick is to modify the distance so as to « push the boundary to infinity »: for an open set $U \subset X$, the distance $d'(x, y) = d(x, y) + |1/d(x, X \setminus U) - 1/d(y, X \setminus U)|$ works.

*EXAMPLE: (1) $\mathbb{R}^n$ is Polish (complete and separable for the Euclidean distance). (2) $C([0, 1])$ with the uniform norm is Polish: the set of polynomials with rational coefficients forms a countable dense subset (Weierstrass), and completeness is classical. (3) $\ell^p$ for $1 \leq p < \infty$ is Polish: sequences with finite support and rational values form a dense subset. (4) The space $\mathbb{R}^{\mathbb{N}}$ with the product topology is Polish (metrisable by $d(x, y) = \sum 2^{-n} \min(|x_n - y_n|, 1)$).*

*REMARK:* The category of Polish spaces is the natural arena for abstract measure theory. Borel sets there are universally measurable, probability measures themselves form a Polish space (under the narrow topology), and the Baire theorem ensures that residual sets are

« large » in the topological sense. It is this conjunction of properties (countability, completeness, well-behaved Borel structure) that makes Polish spaces ubiquitous in probability, functional analysis, and ergodic theory.

Polish spaces thus constitute the natural framework where topology, measure, and countable combinatorics come together harmoniously.

12 Functional Analysis

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*PART XIII — General Algebra*
*PART XIV — Galois Theory*
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*The symbol on the cover is the first letter of the Hebrew alphabet, the aleph. In mathematics, it denotes Cantor's transfinite cardinals. In literature, it is the point in Borges where the entire universe concentrates in a single place. Its name, in the Semitic languages, means "head of an ox." The reader who might see in that head the echo of a horned goddess carrying the sun between her two horizons would not be entirely wrong.*