# Articulated Course

## in

# Mathematics

II

# Licence

From Statement to Space

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## *TO THE READER*

*This course articulates in five volumes and over twelve hundred articles the grammar by which reality composes itself, from the first whole number to the Langlands correspondence, from the brachistochrone to the amplituhedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.*

*Each article develops a single idea, from its motivation to its interpretation. Definitions are framed; theorems are also framed, in red. Proofs are present where they illuminate, sketched where full rigour would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The reader will have no trouble distinguishing the three cases.*

*Ten mathematical objects traverse the collection like red threads: the circle $S^1$, the integers $\mathbb{Z}$, the extension $\mathbb{Q}(\sqrt{2})$, the elliptic curve $y^2 = x^3 - x$, the group $\mathrm{GL}_2$, the function $\zeta(s)$, the space $L^2$, the ring $K[X]$, the symmetric group $\mathfrak{S}_3$, and the torus $T^2$. From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.*

*The figures, numbering six hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colours. They do not replace demonstration; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.*

*The reader may follow the linear path or take the transversal passages between volumes. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands programme; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.*

*Every error reported is an error corrected.*

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# Part I

# Learning to Read Mathematics

# *CHAPTER* I

# *WHAT A MATHEMATICAL STATEMENT IS*

*Before computing anything at all, one must learn to say something that can be judged true or false. This chapter lays the first stone: what is a mathematical statement, and how does it differ from an ordinary sentence? Everyday language tolerates vagueness, implicit meaning, productive ambiguity; mathematics demands that every word be held to account. By learning to formulate, and then to negate, a statement, the reader discovers that rigour is not a constraint: it is the very condition under which thought can advance without stumbling.*

*Articles in this chapter:*

**001.** *When ordinary language falls short — The flexibility of everyday language becomes dangerous when precise reasoning is required.*

**002.** *Saying something that can be judged — Symbols come later; before them, there are statements: sentences one can judge.*

**003.** *What a sentence often lacks — A sentence may be neither absurd nor a statement: it is simply incomplete.*

**004.** *A mathematical truth is not an impression — "one can see that" has no mathematical value: one must be able to prove.*

**005.** *False statement or ill-defined expression — Not all difficulties are on the same level: false $\neq$ ill-defined.*

**006.** *Mathematics begins with distinctions of language — No theorem has been proved in this chapter.*

**007.** *First operation on a statement: negation — One negates neither a command nor an ill-defined expression: one negates a determinate assertion.*

***001. WHEN ORDINARY LANGUAGE FALLS SHORT.***

*The* same word can carry two opposite meanings, and no one notices. In everyday language, we live comfortably with this ambiguity. We fill in what is missing, interpret what is vague, silently correct what does not fit. This flexibility is valuable in ordinary life. It becomes dangerous the moment precise reasoning is required.

*example:* *In a cafeteria, « soup or fruit » usually means one chooses one or the other, not both.*

*example:* *In an administrative rule, « French or scholarship students » includes those who are both French and scholarship holders.* The word « *or* » has not kept the same meaning from one sentence to the next. It is like a thread whose colour shifts with the light: in conversation, context resolves the ambiguity. In mathematics, there is no implicit context to rescue a poorly formulated sentence.

*remark:* Mathematics does not demand nobler words or longer sentences. It demands words that are *held firm*: each term must carry exactly the same meaning wherever it appears. It does not require that everything be complicated. It requires that everything be *decidable*: true or false, well defined or ill defined, proved or unproved.

The first mathematical act is therefore not to calculate. It is to formulate something that can be judged, and that is already an art.

***002. SAYING SOMETHING THAT CAN BE JUDGED.***

*Before* the symbols, before the formulas, there is a more elementary act: uttering a sentence of which one can say whether it is true or false. Many students believe that mathematics consists mainly in manipulating symbols. Symbols come later. Before them, there are *statements*: sentences that can be judged.

Consider five sentences:

$$2+3=5 \qquad ; \qquad 7 \text{ is prime} \qquad ; \qquad x+1>0.$$

« *Close the door.* » « *What will the weather be tomorrow?* »

The first two are mathematical statements in the simplest sense: each asserts something, and that assertion can be judged true or false. The third looks like a statement, but it remains suspended as long as one does not know what $x$ is. The fourth is a command. The fifth is a question. Neither lends itself to the verdict true or false.

DEFINITION 002.1: A *proposition* or *mathematical statement* is a sentence to which one can assign a truth value: true or false.

Mathematical reasoning does not operate on arbitrary sentences. It operates on statements. From them one builds proofs, draws consequences, and identifies errors. Statements are, so to speak, the raw material of reasoning.

REMARK: Other texts sometimes use the word *assertion*. In this volume, we shall mainly use « *statement* », but all three terms designate the same logical object.

The definition is short. Yet it imposes a strict discipline: a vague sentence is not yet a statement, a suggestive sentence is not yet a statement, an incomplete sentence is not yet a statement. Recognising what is a statement and what is not is the first step toward reasoning, and perhaps the most decisive one. For the entire power of logic rests on this single property: a statement can be true or false, and this bivalence is the lever that makes proof possible.

### 003. WHAT A SENTENCE OFTEN LACKS.

*Here* is an expression everyone recognises:

$$x^2 - 1 = 0.$$

It is not absurd. But taken on its own, it is not yet a statement. Does one claim it holds for every real number $x$? That there exists at least one $x$ satisfying it? That one seeks the values of $x$ for which it holds? Without this precision, the sentence hangs in the air, like a question missing its question mark.

By contrast, the following two sentences are genuine statements:

« *There exists a real number x such that* $x^2 - 1 = 0$. »
« *For every real number x, one has* $x^2 - 1 = 0$. »
The first is true: take $x = 1$. The second is false: take $x = 0$ to produce a counterexample. The expression $x^2 - 1 = 0$ is the same in both cases. What changes are the words that frame it, and those words reverse the verdict.

REMARK : In mathematics, the most inconspicuous terms are often the most decisive. « *There exists* » and « *for all* » are not commentary appended to an expression: they fix its scope. The first localises, the second universalises. Without them, the expression $x^2 - 1 = 0$ says nothing one can judge; with them, it becomes a statement whose verdict depends entirely on the word chosen. This opposition between the local and the global runs through all of mathematics: asserting the existence of an object satisfying a condition, or asserting that the condition holds everywhere, are two fundamentally different acts, and it is their distinction that makes reasoning possible.

We shall return at length to these two constructions, which are called *quantifiers*. For now, it suffices to retain this: a sentence may have every appearance of a mathematical statement and yet fail to be one. What it lacks is the frame that fixes its meaning: "for every $x$" or "there exists an $x$." Without this frame, the sentence drifts between true and false without settling anywhere, and reasoning cannot even begin.

***004. A MATHEMATICAL TRUTH IS NOT AN IMPRESSION.***

ONE hears it often: "*it is obvious*", "*one can clearly see that*", "*it must be true*". These phrases sometimes serve a psychological purpose. They carry no mathematical weight. In mathematics, conviction is not a substitute for proof. To say that a statement is true is not to find it plausible. It is either to recognise it as an established result, or to be able to derive it from definitions, hypotheses, and accepted facts.

Three levels must be kept apart, though they are routinely conflated: a sentence that is *plausible*, a sentence that is *true*, and a sentence that is *proved*.

EXAMPLE: *All odd numbers are prime.*

This statement is perfectly well-formed. One can assign it a truth value. It is false. Consider $9 = 3 \times 3$: an odd number that is not prime. The verdict here is swift. But reaching it required exhibiting a counterexample, not trusting an impression.

EXAMPLE: *The sum of two even numbers is even.*

This statement is true. Yet to enter a rigorous argument, one must still know *why* it is true. An elementary proof suffices: if $a = 2m$ and $b = 2n$, then

$$a + b = 2m + 2n = 2(m + n),$$

so that $a+b$ is even. The passage from intuition to proof is like the passage from a rough sketch to an architectural drawing: the aim is not to complicate, but to make visible the reason the statement holds.

REMARK: Mathematical truth is not merely a correct content. It is a correct content that takes its place in a network of reasons: definitions, hypotheses, logical steps. This network is what separates a conviction from a proof, and it is where the entire force of mathematics resides.

Learning to prove is therefore learning to account for one's reasons, and this demand, far from constraining thought, is what gives it reach.

***005. FALSE STATEMENT OR ILL-DEFINED EXPRESSION.***
*A* sentence that fails can fail in two very different ways. Confusing these two failures means understanding neither. A statement can be false: it asserts something precise, but that assertion is incorrect. An expression can also fail upstream, before it even produces a statement: it purports to designate an object, but that designation collapses.

EXAMPLE: *The largest prime number exists.*
This sentence is well-formed. It asserts something clear: there exists a prime number greater than every other. It is a statement. It happens to be false, since Euclid showed that there are infinitely many primes. But at least the sentence says something one can judge.

COUNTEREXAMPLE: *The real number closest to zero without being equal to zero.*
Here the situation is altogether different. This group of words says nothing true or false. It attempts to designate an object, but the designation fails: there is no strictly positive real number smaller than every other strictly positive real number. However close to zero one chooses a real $r > 0$, the real $r/2$ is closer still. The expression therefore fails to pin down an object. It is a verbal mirage: the phrase appears to point at something, but there is nothing at the end of the finger.

REMARK: One must learn to recognise these two situations. In the first case, the sentence is a statement that asserts something incorrect. In the second, the expression does not even manage to pose correctly what it claims to designate. A sharp sentence can be false. A reasonable-looking expression can be ill-defined. Knowing how to distinguish these two failures is one of the first competencies that mathematical reasoning demands, and one of the least taught. Before judging whether a sentence is true or false, one must first verify that it manages to say something at all, and it is this vigilance that opens the way to rigorous definitions.

***006. MATHEMATICS BEGINS WITH DISTINCTIONS OF LANGUAGE.***

*No* theorem has been proved in this chapter. No formula has been derived. Yet something essential has been laid down: the distinctions without which no theorem can be read correctly.

A proposition is not a question. A statement is not a command. A complete sentence is not an incomplete one. A proved truth is not a plausible impression. A well-defined object is not a reasonable-looking expression that fails to designate anything.

*REMARK:* These distinctions seem simple. They are simple in their formulation. They are far less simple in practice, because they foreshadow a deeper principle: in mathematics, the first step of every construction is to fix without ambiguity what one is talking about. Defining a set, stating a theorem, conducting a proof: everything begins with a sentence whose logical status is clear. This is exactly the work that axiomatics will take up on a larger scale: laying down rules of language so precise that reasoning can proceed without appeal to context or intuition.

Mathematics is not exacting out of a taste for complication. It is exacting because it refuses the silent slippages that ordinary language tolerates. The work of this chapter has been to learn to notice those slippages. It is modest work in appearance, but everything that follows rests upon it, as a building rests on its foundations, all the more solid for being invisible.

***007. FIRST OPERATION ON A STATEMENT: NEGATION.***

*ASSERTING* is a first act. The second, equally fundamental, is contradicting. As soon as a statement is well-formed, one can apply a simple logical operation to it: negate it. But one must be precise: one negates neither a command nor an ill-defined expression. One negates a determinate assertion, and one obtains a new determinate assertion.

*PROPOSITION:* The negation of a mathematical statement is again a mathematical statement.

*commented proof:* Let $P$ be a statement. Only two cases arise: $P$ is true or $P$ is false. If $P$ is true, then "*not* $P$" is false. If $P$ is false, then "*not* $P$" is true. In both cases, the sentence "*not* $P$" receives a truth value. It is therefore a statement.
The principle is clear. Putting it into practice, however, holds surprises.

*example:* $2$ *is even. Negation:* $2$ *is not even.*
Here the negation is immediate: one replaces the property by its contrary.

*example:* *There exists an integer $n$ such that $n^2 = 9$. Negation: for every integer $n$, $n^2 \neq 9$.*
To negate the existence of an object satisfying a condition, one asserts that *all* objects fail to satisfy it.

*example:* *For every real number $x$, $x^2 \geq 0$. Negation: there exists a real number $x$ such that $x^2 < 0$.*
The negation of a "*for all*" becomes a "*there exists*". One does not negate this sentence by writing "*for every real number $x$, $x^2 < 0$*": that would assert that *all* squares are negative, which is far stronger than the correct negation.
This last example reveals an essential point: one does not negate a statement by slipping in a "*not*". One must look at the logical structure of the sentence. When a "*for all*" appears, negation transforms it into a "*there exists*", and conversely. Negation is not a mechanical operation on words; it is an operation on structure.

*remark:* The interchange between "*for all*" and "*there exists*" under negation is a first instance of duality: two notions that appear distinct turn out to be linked by an operation transforming one into the other. This symmetry will reappear in De Morgan's laws (the negation of a union is an intersection, and conversely) and, much further on, in the duality between a space and its dual. Recognising such symmetries is one of the guiding threads of all mathematics.

*counterexample:* *"Find a prime number greater than* $10$*."*

This sentence is not a statement: it is a command. One cannot negate it.

Negation is the first tool that logic offers for transforming a statement, and learning to use it correctly is already learning to reason.

*You have just read the first chapter.*

The complete volume contains 401 articles in 84 chapters:


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*The symbol on the cover is the first letter of the Hebrew alphabet, the aleph. In mathematics, it denotes Cantor's transfinite cardinals. In literature, it is the point in Borges where the entire universe concentrates in a single place. Its name, in the Semitic languages, means "head of an ox." The reader who might see in that head the echo of a horned goddess carrying the sun between her two horizons would not be entirely wrong.*