Articulated Course in Mathematics

I

Elements

From Number to Form

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TO THE READER

This course articulates in five volumes and over twelve hundred articles the grammar by which reality composes itself, from the first whole number to the Langlands correspondence, from the brachistochrone to the amplituhedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.

Each article develops a single idea, from its motivation to its interpretation. Definitions are framed; theorems are also framed, in red. Proofs are present where they illuminate, sketched where full rigour would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The reader will have no trouble distinguishing the three cases.

Ten mathematical objects traverse the collection like red threads: the circle S^1*, the integers* $\mathbb{Z}$*, the extension* $\mathbb{Q}(\sqrt{2})$*, the elliptic curve* $y^2 = x^3 - x$*, the group* GL_2*, the function* $\zeta(s)$*, the space* L^2*, the ring* $K[X]$*, the symmetric group* $\mathfrak{S}_3$*, and the torus* T^2*. From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.*

The figures, numbering six hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colours. They do not replace demonstration; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.

The reader may follow the linear path or take the transversal passages between volumes. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands programme; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.

Every error reported is an error corrected.

wdf@clementinium.com

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Language, Computation, and Proof

CHAPTER I

NUMERICAL AND ALGEBRAIC COMPUTATION

Every grammar begins with an alphabet. Before formulating theorems or building theories, one must learn to write: setting the order of operations, handling fractions and powers, recognising the standard identities. These rules are not arbitrary conventions; they are the elementary syntax of a language in which reality lets itself be described. Numerical and algebraic computation is the first word of that language. From the precedence of addition and multiplication to the binomial theorem, this chapter forges the tools from which every mathematical sentence to come will be made.

Articles in this chapter:

001. notation, precedence, numerical computation. *What* is $2 + 3 \times 4$? Nearly everyone, the first time, answers 20. We read left to right, add first, multiply second, and the calculation seems finished. The answer, though, is 14. The mistake is not one of carelessness; it is one of grammar. A computation is not a sentence to be read in sequence. It is a structured expression, and that structure obeys strict rules of precedence.

Why do these rules exist? Without them, a single string of symbols would produce different numbers in different hands. Notation would become ambiguous, and ambiguous notation is useless. The order of operations is the shared convention that resolves this.

definition 001.1: Operations are performed in the following order of precedence:

1. parentheses (starting from the innermost),
2. exponents,
3. multiplications and divisions (left to right),
4. additions and subtractions (left to right).

The logic is hierarchical: the tighter an operation binds its operands, the earlier it executes. Exponentiation binds more tightly than multiplication, which binds more tightly than addition. Think of punctuation: a comma separates less sharply than a semicolon, and a semicolon less sharply than a period. In the same way, $+$ separates less sharply than $\times$, which separates less sharply than an exponent.

example: *Compute* $5+2\times(3^2-1)\div 4$. *First the exponent:* $3^2 = 9$. *Then the parentheses:* $9-1 = 8$. *Next the multiplication:* $2\times 8 = 16$. *Then the division:* $16 \div 4 = 4$. *Finally the addition:* $5 + 4 = 9$. Parentheses are not decoration. They are an *instruction*: “evaluate this first, whatever the usual precedence dictates.”

EXAMPLE: *Compare* $(2+3)\times 4 = 20$ *and* $2+3\times 4 = 14$. *The symbols are nearly identical, but the parentheses change the structure, and therefore the result.*

The fraction bar, too, plays a structural role that is easy to overlook: it acts as invisible parentheses around the numerator and the denominator. Writing $\frac{2+3}{4}$ means $(2+3)\div 4$, not $2+3\div 4$.

COMMON ERROR: Confusing -3^2 with $(-3)^2$. The expression -3^2 means $-(3^2) = -9$: the exponent applies to 3 alone, and the negative sign applies afterward. The expression $(-3)^2$ means $(-3)\times(-3) = 9$: the exponent applies to the number -3 as a whole. A single placement of parentheses flips the sign of the result.

Precision in notation is the first form of mathematical rigour, and the most easily neglected. A line of reasoning can be correct in the mind and false on the page. Respecting precedence, placing parentheses where they belong, writing each intermediate step with care: this is where all serious work begins. Not because the conventions are sacred, but because without them notation stops doing the one thing it was designed for: communicating unambiguously.

002. FRACTIONS, EXPONENTS, ROOTS. *CAN* one add $\frac{2}{3}$ and $\frac{5}{6}$ by simply adding the tops and the bottoms? Does $2^3\times 2^5$ equal 2^{15}? Is $\sqrt{9+16}$ the same as $\sqrt{9}+\sqrt{16}$? In each case the quick answer is tempting and wrong. Fractions, exponents, and roots each follow their own grammar, and that grammar admits no guesswork.

These three families of notation are the building blocks of nearly every algebraic calculation. One learns their rules once; after that, they run in the background of every computation that follows. But the smallest confusion produces errors.

DEFINITION 002.1: A *fraction* is an expression $\frac{a}{b}$ where a is the numerator, b the denominator, and $b \neq 0$. Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $a \times d = b \times c$.

RULE: To add two fractions, bring them to a common denominator. To multiply them, multiply numerators together and denominators together. To divide, multiply by the reciprocal.

EXAMPLE: $\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6} = \frac{3}{2}$. *And* $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \times \frac{8}{9} = \frac{24}{36} = \frac{2}{3}$.

Addition requires common ground: a shared denominator. Multiplication, paradoxically the simpler operation, needs none. But fractions alone are not enough. The moment a calculation involves repetition, we need a notation that compresses many identical factors into a single symbol.

DEFINITION 002.2: For any number $a \neq 0$ and any natural number n, we set $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$. By convention, $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$.

RULE: Exponents satisfy: $a^n \cdot a^m = a^{n+m}$, $\frac{a^n}{a^m} = a^{n-m}$, $(a^n)^m = a^{nm}$.

Multiplying powers of the same base amounts to adding their exponents. The reason is immediate: $a^3 \times a^5$ is three copies of a multiplied by five more, giving eight in total.

EXAMPLE: $2^3 \times 2^5 \div 2^4 = 2^{3+5-4} = 2^4 = 16$. *And* $(3^2)^3 = 3^6 = 729$.

DEFINITION 002.3: For any real number $a \geq 0$, the *square root* $\sqrt{a}$ is the unique non-negative real number whose square equals a: $(\sqrt{a})^2 = a$.

RULE: For $a, b \geq 0$: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ (with $b > 0$). But beware: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ in general.

The square root distributes over products but not over sums. This is one of the most persistent traps in computation: the impulse to split a symbol across a sum is a reflex one must learn to override.

EXAMPLE: $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$. *To rationalize* $\frac{1}{\sqrt{3}}$, *multiply top and bottom by* $\sqrt{3}$*:* $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

A single thread runs through all three families: each one is the reverse of an elementary operation. The fraction $\frac{a}{b}$ undoes multiplication ($b \times \frac{a}{b} = a$). The exponent a^n compresses repeated multiplication into one symbol. The root $\sqrt{a}$ undoes squaring. Every operation has its inverse; every gesture, its return. This architecture of operations paired with their reversals runs through the whole of algebra, from the fractions we meet here all the way to the logarithm, which will one day undo the exponential.

Fractions, exponents, and roots are three voices of a single instrument: the algebraic calculus. Fractions measure proportions, exponents compress repetition, roots undo it. Each has its own strict grammar; but once that grammar is internalized, it becomes the working language of everything that follows.

003. proportionality and the rule of three. *Two* sweets cost one euro, so four cost two. A child sees this without instruction: double the quantity, double the price. The reasoning feels so self-evident that one might take it for a universal law. It is not. Double the temperature of an oven and the cake does not bake in half the time. Proportionality is the simplest model relating two quantities; the real difficulty lies in knowing where it applies and where it deceives.

> *definition 003.1:* Two quantities are *proportional* when one is obtained from the other by multiplying by the same fixed number, called the *constant of proportionality*.

example: If 3 *kg of apples cost* €4.50, *the constant of proportionality is* $\dfrac{4.50}{3} = 1.50$ *€/kg. The price of* 7 *kg is therefore* $7 \times 1.50 = 10.50$ €.

The constant is the single number that binds the two quantities: once it is known, every value on one side determines the corresponding value on the other. Three values suffice to find a fourth; that is the purpose of the rule of three.

rule: The *rule of three* (cross-multiplication) states: if $\dfrac{a}{b} = \dfrac{c}{d}$, then $a \times d = b \times c$. Given any three of the four quantities, the fourth follows.

example: 5 *workers complete a job in* 12 *days. How many days for* 8 *workers? Here the proportionality is* inverse*: more workers, fewer days. The product workers* $\times$ *days is constant:* $5 \times 12 = 60$, *so* 8 *workers take* $60 \div 8 = 7.5$ *days.*

Note the trap: this is *not* a direct application of the rule of three. Inverse proportionality runs against intuition; increasing one factor decreases the other. A student who divides where one should multiply will get the wrong answer. The type of relationship must be identified before the method is applied.