The Letter

Robert Langlands and the Dictionary of Worlds

William de France

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Free sample — foreword, prologue, and chapters 1–2.
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"Nothing is more fruitful — all mathematicians know it — than those obscure analogies, those disturbing reflections of one theory in another; those furtive caresses, those inexplicable disagreements; nothing also gives more pleasure to the researcher."

André Weil,
letter to his sister Simone, 26 March 1940.

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Foreword

This book tells the story of a letter. Seventeen handwritten pages, sent in January 1967 by a young Canadian mathematician, Robert Langlands, to his senior colleague André Weil. The letter proposed, without proof, a precise correspondence between two very different branches of mathematics: the arithmetic of whole numbers on one side, and harmonic analysis on certain groups on the other. What might have remained a mere speculation became, in the decades that followed, a research program of the first importance, to which several generations of mathematicians have given their careers, and one of whose most ambitious versions was proved in 2024.

The wager of this book is that the story can be told without mathematical prerequisites. What one meets along the way is explained as it appears, without skimping on content but also without weighing it down.

At the end of the volume, appendices gather the usual reference tools: a note on the letter itself and its dating, a glossary, a table of symbols, a chronology, biographical notes on the main figures, an annotated bibliography, and an index of names and concepts.

The reader who wishes, once this book is read, to find a more technical and more detailed exposition of the program will find it in the companion volume Langlands — The Secret Unity of Mathematics *(2024), from the same publisher, where the precise statements, the proofs, and the mathematical apparatus are set out in the discipline's own register.*

WdF

Prologue

Princeton, January 1967

Picture a university office on a winter afternoon. The walls are pale wood, the window looks out onto a park white with snow. On the table, a ruled writing pad, a fountain pen, a cup of tea going cold. A man in his thirties writes slowly, stops, rereads what he has just set down, crosses it out, starts again. He has been working on this letter for several days. When he is finished, it will run to seventeen pages.

The office belongs to Robert Langlands, in the Fine Hall building at Princeton University. Langlands is Canadian, born in British Columbia, the son of a carpenter. He discovered mathematics almost by accident at a small provincial university and never looked away. He arrived at Princeton seven years earlier and now holds a position as *assistant professor*. At thirty, he is unknown outside a small circle of specialists; but he has a curiosity that runs well beyond his thesis topic. Several years of reading work that few people read together have left, in his mind, an intuition that has gradually taken shape.

The letter is addressed to André Weil, who works a few hundred metres away, in another institute at Princeton. Weil is sixty, with an international career behind him and the unofficial status of patriarch of number theory. The previous summer, at the International Congress of Mathematicians in Moscow, he gave a lecture that people are still talking about. He is not known for patience with vague ideas.

Langlands is not quite sure of his ground. In the first sentence of the letter, he almost apologizes. He writes, in substance, that if his correspondent is willing to read it as pure speculation he would be grateful, and that otherwise he is sure a wastepaper basket is close at hand. That sentence has become one of the most celebrated in the history of modern mathematics.

The seventeen pages that follow do not resemble what his contemporaries usually read. It is not an article, nor an isolated conjecture, nor a proof. It is a *program*. An overall plan for an entire branch of mathematics yet to come. A prediction about the shape of theorems that do not yet exist and that will, he supposes, take decades to prove.

The idea is this. Over the twentieth century, mathematics developed two large regions that grew up almost independently. The first is number theory. It studies the integers, the primes, equations with integer coefficients, and the symmetries that permute their solutions. It is a very old region, with familiar concerns. The second is called harmonic analysis. It studies functions living on spaces with many continuous symmetries, and the way those functions decompose. It is a younger, more abstract region, born in the twentieth century from the need to study waves, oscillations, spaces of rotation.

At first glance, these two regions have nothing in common. One is discrete, algebraic, turned toward arithmetic. The other is continuous, analytic, closer to physics than to number theory. One can spend a career in either without ever meeting the other. Many mathematicians do.

Langlands's intuition is that they speak to each other. More than that: that they are, in a precise sense, two descriptions of one and the same reality. Each arithmetic object should correspond, term for term, to an analytic one. Each question posed in one language should be re-expressible in the other. A dictionary, in short, should exist between these two regions: not a dictionary in the loose sense where two things "resemble" each other, but a precise dictionary, where every word in one language has an exact counterpart in the other.

The idea is not entirely new. Early in the twentieth century, there was already a theory that realized a dictionary of this kind in a very simple case. It is called *class field theory*. It took fifty years of work by first-rank mathematicians (David Hilbert, Teiji Takagi, Emil Artin, Claude Chevalley) to reach its finished form, and it handled only the simplest possible case of the dictionary. What Langlands proposes, in his seventeen pages, is to generalize this theory to every case. To every dimension. To every group. Without proof. With only the intuition that such a structure, however vast, cannot fail to exist.

Weil will read the letter. He will not throw it away. He will reply briefly, politely, with measured skepticism. But he will pass it around. In the years that follow, Langlands's intuition will become a program, *the Langlands program*, to which several generations of mathematicians will devote their careers. Seminars will meet in Berkeley, Oxford, Paris, Moscow, Bonn. Decades will pass. Pieces of the dictionary will be proved. Fermat's Last Theorem, open since 1637, will fall, almost incidentally, as a consequence. Fields Medals will be awarded to those who make a step on the path.

And one day in July 2024, fifty-seven years after the letter, a team of five mathematicians led by Dennis Gaitsgory, at Harvard, will announce the complete proof of one of the most ambitious versions of the program. The result will fill five main papers and more than a thousand pages.

This book tells what was in the letter of 1967, and what it took for the letter to be right. It assumes no particular mathematical background. All it asks is a little patience, and the desire to watch an idea unfold over time.

Act I

The Two Shores

I

The symmetries of numbers

The equation $x^2 = 2$ has two solutions, which we write $\sqrt{2}$ and $-\sqrt{2}$, roughly equal to 1.414 and -1.414. A slightly strange question suggests itself about them: which of the two, more than the other, deserves to be called $\sqrt{2}$? Which is the "real" one? By convention we give the name to the positive one. But is that convention justified by anything mathematical?

On reflection, no. Nothing whatsoever distinguishes $\sqrt{2}$ from $-\sqrt{2}$ algebraically. Every relation satisfied by one is satisfied by the other. Every polynomial with integer coefficients that vanishes at $\sqrt{2}$ also vanishes at $-\sqrt{2}$. The two values are interchangeable: one can replace either by the other in any algebraic formula without anything changing.

This small observation, apparently banal, contains the seed of an idea that will upend algebra in the nineteenth century. What matters in an equation is not the solutions themselves. It is the *symmetries* that permute them.

The idea was carried to its full maturity by a young French mathematician who did not live to see it bear fruit. Évariste Galois died at the age of twenty, in May 1832, of a gunshot wound received in a duel whose precise cause is still not known. The last years of his life he spent largely in prison on political charges. On the nights before his duel, sensing perhaps what was coming, he hurriedly wrote out his mathematical ideas in a manuscript left to a friend, along with a letter that begins with the famous words "I have no time." It took several decades before mathematicians fully grasped what he had discovered.

The question that occupied Galois had occupied algebraists for the previous two centuries. It is stated simply. Given a polynomial equation, can one write its roots, starting from the coefficients, using only a finite sequence of elementary operations: addition, subtraction, multiplication, division, and the extraction of roots (square, cube, fourth, fifth, and so on)?

For equations of the first degree, of the form $ax + b = 0$, the answer is obviously yes: the solution is $x = -b/a$. For equations of the second degree, of the form $ax^2 + bx + c = 0$, the answer is again yes: the formula has been known since the Babylonians, two thousand years before our era, and is in every school textbook. For the third degree, the formula was discovered by the Italian Renaissance algebraists, Cardano, Tartaglia and Ferrari above all, after the famous public contests whose echoes fill the histories of mathematics. For the fourth degree, a formula exists too, though more complicated. But for the fifth degree, no one could find one, despite two centuries of work by Europe's best mathematicians.

Galois's answer to the question was not a new formula. It was something radically different: a change of viewpoint. Instead of looking for the formula directly, Galois proposed to look at the *permutations* of the roots, that is, the ways one can interchange the solutions while preserving their algebraic relations.

To each equation, then, corresponds a set of permitted permutations. This set has a particular structure: one can combine two permutations by carrying them out one after the other, and the result is again a permitted permutation. One then speaks of a *group*, and the group attached to an equation is called the *Galois group* of that equation.

Take our equation $x^2 = 2$ again. The roots are $\sqrt{2}$ and $-\sqrt{2}$. The possible permutations are two in number: the identity, which leaves each root in place, and the exchange, which swaps them. The Galois group of this equation therefore has two elements. Its structure fits on a single line.

For a general equation of the second degree, the Galois group is, as a rule, a group of two elements. For an equation of the third degree, it can have up to six elements. For one of the fourth, up to twenty-four. For one of the fifth, up to a hundred and twenty. As the degree grows, the maximum size of the group grows very fast. But size is not the decisive criterion. What matters is the *internal structure* of the group.

The precise statement Galois proved, which settled the question that had been open for two centuries, is this. An equation admits a formula for its roots in terms of radicals if, and only if, its Galois group has a certain structure that Galois called *solvable*. Solvable groups are the ones that can be taken apart, layer by layer, into simple pieces (the abelian pieces, that is, the commutative ones, in which the order of combination does not matter). Groups that are not solvable cannot be taken apart in this way, and for them no formula in radicals exists, however clever.

The distinction between solvable and unsolvable equations, therefore, does not depend on the ingenuity of mathematicians. It is inscribed in the equation itself, through the structure of its Galois group. For equations of degree one, two, three, four, the Galois group can always be decomposed into simple pieces. For degree five, this is no longer true in general: the Galois group contains, at bottom, a piece that can no longer be decomposed, and its presence is enough to prevent any formula.

That is for a single equation. But Galois's idea has a reach that extends well beyond this initial case. It says, in substance, that to understand an algebraic problem, one must look not at the solutions themselves, but at the symmetries that permute them. This intuition is one of the most fruitful in the history of mathematics. It will serve, throughout the twentieth century, as the guiding thread of an almost complete reworking of algebra.

And here the story starts to concern us directly. Galois had applied his idea to *one* equation at a time. His successors will apply it to *all equations at once*.

Suppose that, instead of a particular polynomial, one takes the whole collection of polynomials with rational coefficients. Suppose one solves every one of them. And suppose one gathers all the solutions so obtained into a single large collection. That collection is enormous. It contains $\sqrt{2}$, of course, but also $\sqrt[3]{7}$, $\sqrt[5]{11}$, and countless other numbers, provided they are roots of some polynomial with rational coefficients. Such numbers are called *algebraic numbers*. They stand in contrast to the numbers that cannot be obtained in this way, called *transcendental numbers* (the most famous are π and e).

The collection of all algebraic numbers has a structure of its own: it forms what one calls the *algebraic closure* of the rational numbers, written $\overline{\mathbb{Q}}$. It contains the rationals (which are trivially algebraic, since p/q is a root of the polynomial $qx - p$), and all the numbers one can obtain from them by solving polynomial equations. Everything that polynomial algebra can reach, starting from the rationals, is there.

Now let us apply Galois's idea, no longer to a particular polynomial, but to the whole collection $\overline{\mathbb{Q}}$ viewed as an extension of the rationals. The symmetries of $\overline{\mathbb{Q}}$ that fix each rational number (that is, that leave $0, 1, 2, 1/3$ and every other rational in place, while permitting permutations among the algebraic non-rational numbers) form a group. This group is called the *absolute Galois group* of the rationals, and is written $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

It is a colossal object. Unlike the Galois group of a particular polynomial, it is not finite. It contains the Galois group of every polynomial equation, all nested within one another in a structure one can in principle describe, but which remains, in practice, impenetrable. It is, in short, the group that gathers all the arithmetic information available about the rational numbers.

For this is the astonishing claim that will accompany us throughout this book: that all the arithmetic information one can state about the rational numbers is, in one form or another, inscribed in the structure of this group. The way a prime number decomposes in a given extension. The existence or non-existence of integer solutions to a given equation. The reasons certain congruences between seemingly unrelated numbers are inevitable. All these questions have their answers, in principle, somewhere in the structure of the absolute Galois group.

In principle. Because there is a sizeable obstacle: this group is too large to be looked at directly. It is infinite, it has no simple explicit description, and one knows its internal structure only through theorems about its finite fragments. To try to describe it directly would be like trying to look at the sun without a filter: possible in principle, impracticable in practice.

So one needs an instrument. An indirect way of extracting from this group fragments of information that, together, reconstitute what it contains. That instrument exists. It was developed from the 1890s on, not for the absolute Galois group, but for finite groups. It is called a *representation*.

2

Representations

One of the most fruitful ideas in modern algebra fits in a single phrase: to study a group, one turns it into matrices.

A matrix, for recall, is an array of numbers arranged in rows and columns. A matrix of size 2×2, for example, is a square of four numbers. One can add two matrices, one can multiply them, and multiplication obeys rules close to those of ordinary numbers, with one difference: when one multiplies two matrices A and B, the order matters. The product AB is not, in general, equal to BA.

It is precisely this last property that makes matrices interesting for our subject. For the groups one wants to study, beginning with the Galois groups, are not commutative either. Combining two symmetries in one order or the other can give different results. Matrices, in their multiplication, reproduce that property. They therefore offer a natural setting in which to translate groups into objects one can calculate with.

The idea was formulated in the 1890s, at the University of Berlin, by a German mathematician named Ferdinand Georg Frobenius. He was answering a question raised by his colleague Richard Dedekind, and in thinking it over he realized that there was a systematic way of turning any finite group into a family of matrices. The rule is simple. To each element g of the group one assigns a square invertible matrix $\rho(g)$ of size $n \times n$. One asks only one thing of the assignment: if one combines two elements g and h in the group, the product gh must correspond to the product $\rho(g)\rho(h)$ of the associated matrices. In other words, the group's law of combination must be respected by the matrices.

Such an assignment is called a *representation* of the group. The integer n is called the *dimension* of the representation. The larger n, the larger the matrices, and the more informative the representation can be.

Why does this change anything? Because once the group has been turned into matrices, one can calculate. Matrices have eigenvalues, a trace (the sum of the diagonal entries), a determinant, a characteristic polynomial. The whole machinery of linear algebra, developed over two centuries to study systems of linear equations and geometry, becomes available. A group that was only a combination table, often impossible to write out completely, now sits on a vector space where its action becomes visible. One can examine it, compare it, decompose it.

The most informative representations are those that cannot be decomposed further. What is meant by that is simple. Suppose a representation sends the elements of the group into $n \times n$ matrices. Suppose also that there is, inside the n-dimensional space, a smaller subspace that every one of those matrices leaves invariant (sends to itself). Then one can restrict to that subspace and obtain a smaller representation, of lower dimension. A representation that has no nontrivial invariant subspace is called *irreducible*. It is, in a sense, an atom of the theory: it cannot be broken into simpler pieces.

Frobenius, and after him Issai Schur and Richard Brauer, proved on these irreducible representations a series of identities of a cleanness rare in algebra. The number of irreducible representations of a finite group is exactly equal to the number of its conjugacy classes (a notion we shall not detail here, but which depends only on the structure of the group). The sum of the squares of the dimensions of the irreducible representations equals the order of the group (that is, its total number of elements). The full information of the group fits into a small table called the *character table*, which essentially encodes everything there is to know. For finite groups this theory is rare in its efficiency: it reduces the study of a group to that of its character table, and the interesting calculations all proceed from that table.

But the group Langlands cared about, the absolute Galois group of the rationals that we met in the previous chapter, is not finite. It is infinite. It even has a particular structure: it is what is called *profinite*, which means, without going into detail, that it is the limit of an infinite sequence of finite groups nested within one another, like Russian dolls. To study such a group by means of representations, one has to adapt Frobenius's framework.

Two modifications suffice. First, one no longer takes matrices with coefficients in just any numbers. One takes them with coefficients in a particular type of number, called the *p-adic fields*. This is a mathematical construction fascinating in its own right, and deserving of a book of its own, but for which it is enough here to retain the general idea. The *p*-adic fields are numbers in which the notion of closeness works differently from what school taught us. In ordinary numbers, two values are close if their difference is small. In the *p*-adic fields, two values are close if their difference is divisible by a large power of a prime *p*. This strange way of measuring closeness happens to be exactly the one that Galois representations need in order to capture the arithmetic structure of the rational numbers.

Second, one requires the representation to be *continuous*, which means, intuitively, that it has no sudden jumps. If two elements of the Galois group are very close (in the particular sense of the profinite topology we just mentioned), their images under the representation must be close as well (in the *p*-adic sense). This continuity condition is what forces the representation to respect the internal structure of the group, instead of simply gliding over its elements.

Such a representation is called a *p-adic Galois representation*. It is one of the two types of objects Langlands will put into correspondence in his letter. It is both concrete (they are matrices, one can compute with them) and intimately tied to arithmetic (they encode information about the rational numbers that would otherwise be inaccessible).

But how exactly? To see why these representations are so useful, one has to describe a particular type of element of the Galois group, called the *Frobenius*.

For each prime p, there exists within the absolute Galois group a particular element (or more precisely, a class of elements), called the Frobenius at p and written Frob_p. It is defined by its action on numbers considered modulo p. Working modulo p means not distinguishing two integers that differ by a multiple of p: modulo 7, the numbers 3, 10, 17, 24 are all considered the same, since they all differ by multiples of 7. One obtains in this way a system of numbers smaller and simpler than the ordinary integers. The Frobenius at p is the symmetry which, in that system, sends each element x to x^p. This operation preserves all the algebraic relations that hold modulo p: it is indeed a symmetry.

The Frobenius matters because it condenses, in a single element, all the arithmetic information tied to the prime p. And when one applies a Galois representation to that element, one obtains a matrix. That matrix, like any matrix, has a characteristic polynomial, a trace, a determinant, eigenvalues. These quantities are finite, concrete, and computable in principle. They are the *local data* attached to the prime p by the representation.

Here is the observation. If one gathers these local data for every prime p (for $p = 2, 3, 5, 7, 11$, and so on), one obtains a sequence of matrices, or more simply a sequence of characteristic polynomials, or more simply still a sequence of numbers (the traces of the Frobenius matrices). This sequence is the *imprint* of the representation: a collection of information, prime by prime, that almost completely characterizes it.

In practice, two distinct representations give distinct imprints. The local information at each prime, taken together, determines the representation. And it is this imprint, this collection of information prime by prime, that the L-function (which we shall meet two chapters from now) will condense into a single analytic object.

But before reaching the *L*-function, we have to meet the second half of the picture. The Galois representations are only one of the two shores that Langlands wants to connect. The other shore is that of *automorphic forms*. They arose in the mathematics of the twentieth century along a completely different road, discovered by mathematicians who were not thinking of number theory at all, and their beauty, on first acquaintance, has something almost unaccountable about it.

You have just read the foreword, the prologue, and the first two chapters.

The complete book contains a prologue, twelve chapters in three acts,
an epilogue, and seven appendices.

Act I — The Two Shores

The symmetries of numbers · Representations · Automorphic forms · L-functions

Act II — The Letter

Princeton, January 1967 · The dictionary · Functoriality, local, global

Act III — What the Letter Held

Class field theory · Modularity and Fermat · The trace formula and the fundamental lemma · Geometric Langlands · 2024

Epilogue — Oslo 2018, Cambridge 2024.

Appendices:

Note on the 1967 letter · Glossary · Table of symbols · Chronology · Biographical notes · Annotated bibliography · Index of names and concepts

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