The Apocalypse According to Galois

A True Story
in Three Unveilings

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Éditions Clementinium

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”Mathematics is the art of giving the same name to different things.”

Henri Poincaré

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Prologue

The candle had only an inch of wax left. The young man pushed it toward the edge of the table, away from the sheets of paper, and took up his pen. He was twenty years old. In a few hours, at daybreak, he would fight a duel. He had known it since the night before, and he wrote like a man who did not expect to return.

What he wrote that night was not only letters of farewell, though he also traced those, feverishly, to his republican friends. The essential was mathematics. Notes scribbled in the margins of memoirs that no one had wanted to read. Hurried references to results that he alone in the world knew. He wrote to his friend Auguste Chevalier, the only one to whom he could entrust what mattered most. And in the margin of a draft, with a hand running too fast, he scribbled these words: “I have no time.”

What he saw, in that room where the flame was dying, was not one more formula. It was a different way of posing the question. He saw that to each equation corresponds a play of hidden symmetries, a way of swapping its solutions without breaking anything. And he saw that the shape of this play, its secret grammar alone decides whether the equation can be solved or whether it resists forever. Impossibility, he understood, is not a blind wall erected before us by chance. It is the sign that an architecture is there, underneath, waiting for us to look at it.

He saw even further. He guessed that this architecture extended far beyond the small world of equations, that it reached into territories whose outlines he could only glimpse, in fatigue and fear. He did not have the time to explore them. No one, for a century, had the time to even follow him.

Évariste Galois died on May 31, 1832, two days after that night, from a bullet received in the early morning. Eleven years later, a mathematician named Joseph Liouville pulled his manuscripts from a drawer, truly read them, and understood they were right. The world took a century to unfold them.

This book tells the story of that unfolding. To do so, we must say a word about a word. The Greek *apokalypsis*, from which we get "apocalypse," does not mean a catastrophe, nor the end of the world, nor a great crash. It means an *unveiling*: the act of removing a veil from something that was there, present all along, but which no one saw. Nothing is destroyed. Something simply becomes visible.

The story that follows is that of three of these unveilings, told in the order they occurred. The first is that of Galois: the discovery that certain equations resist any possible formula, not through our clumsiness but by their very nature, and the hidden symmetry that explains why. The second is that of Langlands: a dictionary between mathematical continents that everything seemed to separate, linking algebra to physics, to geometry, to the shape of the world. The third is that of the moderns, almost before our eyes: the proof, in 2024, that they speak the same language at heart.

This story begins long before the young man with the candle. It begins four thousand years earlier, in the heat of a Mesopotamian workshop, and it does not end with him: it continues today, in seminars where we are still unfolding the veil he lifted with one hand.

A promise, before entering. You will have no equations to solve. No formula will be asked of you, no symbol imposed. You will need only one thing, the very thing Galois cruelly lacked that night: a little time, and a little attention. The rest is men, dates, poorly heated rooms, letters arriving too late. And an idea of beauty you will never forget.

The story begins long before him, with scribes who engraved recipes in clay.

Part I

The Impossible

1

Recipes and Duels

Around two thousand years before the common era, in a Mesopotamian city whose name has been lost, a scribe pressed his reed stylus into a tablet of fresh clay. It was hot, undoubtedly. The clay took the imprint of the cut reed, and the scribe wrote down a recipe. Not a recipe for cooking, nor a prayer: a procedure for calculation. Someone had posed a small problem to him, and he recorded the solution, step by step, so that others after him would know how to do it again.

The problem has come down to us. It is engraved on a tablet kept today at Yale University under the unpoetic name of YBC 6967. It asks this: find two numbers whose product is sixty and whose difference is seven. The scribe did not possess our language. He spoke neither of unknowns, nor of equations, nor of that famous "x" that school makes us search for. He stated everything in words, Akkadian words whose exact translation scholars still debate. But the procedure he described, made of a few elementary gestures, is almost exactly what schoolchildren around the world still learn today. Four thousand years later, his recipe still works.

Here we must pause on a word that we will meet constantly: *solving*. For a mathematician, solving an equation does not mean guessing its answer, nor approximating it as closely as one wishes with a calculator. It means writing the solution by means of a finite recipe, starting from the data of the problem, and using only five allowed gestures. Four of them are the operations from school: adding, subtracting, multiplying, dividing. The fifth is root extraction: square root, cube root, and so on.

These five gestures are not equal. As long as one sticks to the first four, one goes in circles in the world from which one started: one enters with ordinary fractions, one comes out with ordinary fractions. Nothing new. Only root extraction manufactures truly new numbers, numbers that did not inhabit the starting world. It is root extraction that opens a door.

We can imagine this quest as the ascent of a mountain. Each degree of an equation is a camp higher on the slope. The first camp, the second degree, was conquered very early: this is what the scribe was already doing without knowing it. Above, the path grew steep, and no one knew how far it climbed.

For three thousand years, the summit remained the same. The Greeks took up these calculations and translated them into their own language, that of lengths and areas: where the scribe added numbers, Euclid constructed segments with a ruler and compass. The result was the same, but no symbol was ever traced; geometry took the place of algebra. Then, in Baghdad, in the ninth century, a scholar named al-Khwarizmi wrote a treatise whose title bequeathed to all the languages of the world a word you pronounce without thinking about it, the word *algebra*. In it, he classified second-degree equations into families, six in all, and gave for each a reliable method, applicable without thinking. For the first time, solving became an organized art, with its rules, transmissible to anyone.

But the second degree remained the ceiling. The mathematicians of India, and those of medieval Europe, refined the recipes, transmitted them, and taught them, without ever crossing the threshold above. Three thousand five hundred years separate the scribe of Mesopotamia from the continuation of this story, and during all this time confidence remained intact: if the second degree had finally yielded its formula, the third would yield its own. It was just a matter of searching long enough, with enough cleverness. People were sure of it.

The breakthrough came from Italy, and it came with a crash.

In Venice, in February 1535, a hall filled with onlookers gathered to witness a duel. Not a sword duel, but a duel of mathematicians, which, in the Renaissance, drew crowds and made reputations. Each had handed the other a list of thirty problems. Whoever solved the most in the allotted time would win the money, the banquet, and the glory.

On one side, Antonio Maria Fior, heir to a secret method that a deceased master, Scipione del Ferro, had passed on to him on his deathbed. On the other, Niccolò Fontana, nicknamed Tartaglia, meaning "the stutterer." As a child, French soldiers had slashed his face with a saber, leaving him with hesitant speech and a scar he hid under his beard. Self-taught, poor, he had learned alone. Fior believed him harmless. He was wrong.

In the days preceding the contest, Tartaglia had found, through sleepless nights, a general method for the third degree. When morning came, he solved Fior's thirty problems in two hours. Fior, for his part, solved none. Thirty to zero. The room saw a stuttering man crush an opponent who believed himself unbeatable.

Tartaglia kept his treasure to himself: a secret of this kind was worth a university chair. He kept it, at least, until a man relieved him of it. Gerolamo Cardano, a renowned physician, astrologer, inveterate gambler, and one of the most brilliant and devious minds of the century, pressed him for months and finally wrested the method from him in 1539, under the seal of an oath: he would never publish it. Cardano swore. Then Cardano published.

In 1545, his great book, the *Ars Magna*, appeared. The method of the third degree was in it, complete, exposed to the whole world, with, it must be said, the honest mention of del Ferro and Tartaglia as inventors. This did not console Tartaglia, who cried betrayal and spent the last twelve years of his life poisoned by this quarrel. The affair was human, petty, full of rancor. The formula, however, was magnificent.

Magnificent, and disturbing. It can be described without writing it down: two nested cube roots, each hiding a square root beneath it. Already heavy. But the most troubling thing was not its length. It was its trap. In certain cases where the equation nevertheless possesses three perfectly ordinary solutions, solutions that one can point to with a finger on a drawing, the recipe required, along the way, extracting the square root of a negative number, something that had no meaning at the time. Cardano noted the phenomenon with a perplexity tinged with annoyance. These ghosts would be baptized much later as "imaginary" numbers. The calculation crossed the impossible and returned with the right answer, intact. It was as if, to reach a real treasure, one had to take a tunnel that passes under a mountain that does not exist.

The fourth degree fell almost immediately after. Lodovico Ferrari, Cardano's pupil, barely eighteen years old, found in 1540 the way to reduce a fourth-degree equation to a third-degree equation, which one now knew how to handle. The formula that came out of it was a monster: roots nested in roots, a whole page of calculation. But the principle held. Cardano published it, too, in the *Ars Magna*.

In three-quarters of a century, Italian algebra had taken two fortresses that three thousand years of mathematics had left untouched. The third degree in a decade, the fourth immediately after. The slope seemed gentle, the march irresistible. The fifth would fall in its turn; it was only a matter of time, effort, and the right angle of attack. Euler himself, the great calculator of the following century, would try it and fail. No one concluded anything from this.

And there, in the jubilation of these victories, something went unnoticed. The second degree fit on one line. The third occupied a paragraph. The fourth filled a page. The difficulty did not add up from one degree to another: it multiplied. The roots nested in each other like Russian dolls, each layer hiding one more. It was not a lengthening: it was a change of nature. Each new floor did not add a step to the staircase. It added an entire building, and its plan became more complicated as one climbed. No one knew how to read this warning. People saw in it a slightly higher step, nothing more. It was a wall beginning to rise, and they took it for a slope.

The third and fourth degrees had fallen in the clash of duels. There remained the fifth. No one guessed that the mountain, this time, was a wall.

2

The Wall

In Modena, in 1799, a man in his fifties published two thick volumes at his own expense. His name was Paolo Ruffini. He was not a mathematician by trade: he was a doctor, a clinical professor at the university, and he taught mathematics on the side, as much out of love for it as out of duty. His book was entitled *Teoria generale delle equazioni*, and it carried a conviction that no one wanted to hear. The fifth degree, Ruffini wrote, cannot be solved. There is no general formula. It is not a matter of searching better. There is nothing to find.

Let us measure how extraordinary this statement was. For three millennia, it had been believed that to each equation there must correspond its formula, that it was just a matter of flushing it out. Habit had ended up welding a tacit equivalence in minds: equation equals formula, and one searches. To question this was not to announce one more failure; it was to change the meaning of the word "solving." For to say "there is none" is not to say "I did not succeed." It is to assert that no human being, however brilliant, in a hundred years or a thousand, will ever succeed, and this for a simple reason: the thing does not exist.

But how do you prove that a thing does not exist? One cannot try all possible formulas and find that none work: there are an infinite number of them, and one would never finish. An argument of a completely different nature is needed, one that no longer speaks of calculation but of *structure*: showing that the very shape of the desired object contradicts itself, that a formula for the fifth degree, if one were to imagine one, would have to be both this and its opposite. This is no longer shopkeeper's algebra. It is almost philosophy, done with the rigor of mathematics.

Ruffini sent his two volumes to Lagrange, the greatest living mathematician, the only man he thought would understand him. Lagrange did not reply. His silence was probably not contempt: it was embarrassment. Ruffini's proof ran to hundreds of pages and, in places, the reasoning skipped steps. The rare readers noted these gaps without being able to say if they could be filled. Ruffini spent years patching it up, without ever convincing anyone. In 1821, Cauchy, the only front-rank scholar to have written to him, sent him a polite letter praising "the importance of what you have *sought* to establish." The word said it all: sought, not established. Ruffini died the following year, in 1822, without having been believed. He was right, and he died without ever really knowing it.

The proof, the real one, came from the cold. Niels Henrik Abel was born in 1802 in a village in southern Norway, into a poor family. His father, a pastor, was sinking into alcohol; he died when Abel was eighteen, leaving his family with nothing. The young man survived on meager scholarships and a few teachers who had sensed his genius. In 1824, he drafted a memoir. Six pages. He had to pay for it out of his own pocket, and to reduce the printer's bill, he compressed his text to the point of rendering it almost illegible. In these six cramped pages, he demonstrated, rigorously, what Ruffini had not been able to make indisputable: the general equation of the fifth degree has no solution by radicals.

His idea can be described without a formula. Abel reasoned by contradiction, that is, by pretending to believe the opposite. Suppose, he said, that a formula exists. Then it would have to obey certain symmetries, certain ways of exchanging the solutions without breaking anything. But these symmetries, starting from the fifth degree, demand contradictory things of one another: they require one another to be both this and its opposite. A recipe that would have to satisfy all these requirements at the same time cannot therefore be written. Not by bad luck: by internal contradiction. The door is not locked. There is no door.

Abel met the same fate as Ruffini: silence. He sent his memoir to Gauss, in Göttingen, the peak of European science. Gauss did not read it: the text was too compressed, due to those famous printer's economies. Abel also addressed another work, on a related subject, to the Académie des sciences in Paris. Cauchy and Legendre were charged with examining it. No report appeared. The manuscript was quite simply lost, and was found years later, by chance, among Cauchy's papers.

Abel finally obtained a travel scholarship. He traveled through Germany, France, and Italy. He became close with the young Jacobi. He knocked on doors. None opened to a position. Discouraged, he returned to Norway, lived by private tutoring, and contracted tuberculosis. The winter was harsh. He worked constantly, writing memoirs that, on their own, would have sufficed to make him famous. His condition deteriorated rapidly. He died on April 6, 1829, in Froland, at the home of his fiancée's family. He was twenty-six years old.

Two days later, a letter arrived at his address. It came from Berlin. He was offered a professorship at the university. Two days. Recognition had rolled toward him while disease rolled faster, and it crossed his threshold when he could no longer read it.

Abel's theorem was final. But it left unanswered the most beautiful question, and all the rest of this book lies in this "but." Abel had proved that there is no *general* formula, a single formula valid for all equations of the fifth degree at once. Yet certain specific fifth-degree equations can be solved quite easily. Take the equation $x^5 = 2$, which asks what number, taken five times as a factor, gives two: its solution is simply a fifth root, a single, clean, neat extraction. This equation yields. Other equations of degree five, on the other hand, resist. Abel knew that some yield and others hold. He had no way of looking at a given equation and saying *which*. He pronounced a verdict on the crowd; he did not know how to judge the individual.

Here is where the decisive shift occurs. The question ceases to be: "how do we solve this equation?" It becomes: "what is it, in this equation, that decides whether it is solvable or not?" We no longer search for an answer; we search for the reasons for an answer. We no longer force the door; we study the architecture of the building in which this door is located. This displacement of the gaze, from the recipe toward the cause, is the gesture that will return at each major turning point in this narrative.

A trail existed, laid down thirty years earlier. In 1770, in his *Réflexions sur la résolution algébrique des équations*, Lagrange had dissected the methods that worked for degrees two, three, and four. He had noticed that all of them, under their different disguises, rested on the same thing: one looks at what happens when one exchanges the solutions with one another. Certain combinations remain intact under these exchanges, others do not, and this difference carried the decisive information. Lagrange had laid the foundations. He had not built the house. He had laid the right stones and stopped at the threshold.

This missing criterion, the one that would change a statement of impossibility into an instrument of understanding, a Parisian teenager was going to find, by reading Lagrange between the lines.

3

The Land Where Roots Live

It is said that a man died for telling the truth about a square. His name was Hippasus, he was from Metapontum, and he belonged to the Pythagorean brotherhood, philosophers who had built their entire worldview on a certainty: all is number, and every number is a ratio of two integers, a fraction. Hippasus is said to have demonstrated that the diagonal of a square of side one escapes this law. This diagonal has a very real length, one can trace it with a ruler, but no fraction, none at all, gives it exactly. Legend has it that his brothers threw him into the sea for disclosing a secret that ruined their doctrine.

The anecdote is probably false. The Pythagoreans probably drowned no one, and it is doubtful that Hippasus ended up at the bottom of the Aegean Sea over a matter of geometry. But the fact itself is indisputable, and it is one of the oldest shocks in the history of ideas: there exist numbers that are not fractions. Not "difficult to write as a fraction." No fraction at all. For men who had sworn that all was a ratio of integers, it was like discovering a continent on a map they believed to be complete.

Let us take it by its purest case. Consider the equation $x^2 = 2$, which asks what number, multiplied by itself, gives two. This number is exactly the length of the diagonal of our square of side one. One can approximate it as closely as one wishes (one point four, one point four one four two, and so on), but the decimals run on without end and without ever repeating. No pattern, no periodicity, ever.

The proof that this diagonal is not a fraction takes only a few lines, and it can be stated in words. Suppose that it is written as a reduced fraction, simplified as much as possible. A simple argument then shows that its numerator would have to be even; and from there, that its denominator would have to be even too. But a fraction whose two terms are even could be simplified further: we had therefore supposed it reduced in error. The assumption bites its own tail. The diagonal of the square lives somewhere other than in the land of fractions.

To understand what follows, one must look at the land of fractions as it is: an astonishingly comfortable, and closed, territory. Take two fractions, add them: you obtain a fraction. Subtract, multiply, divide (without dividing by zero): always a fraction. Never does one of these four operations make you cross the border. One can spend a whole life of calculation there without ever leaving. Mathematicians would say things differently, but one may picture fractions as a world closed in on itself, self-sufficient, perfectly peaceful.

This closure is not a detail; it is a rare virtue. A world where the four operations never eject you is a world where one can calculate in confidence, without fearing a sudden fall into the void. It is precisely this comfort that the diagonal of the square comes to disrupt.

The diagonal breaks into this world. It is not there, yet the equation demands it, requires it as a solution. What to do? We do not give up, and we do not cheat. We enlarge the world. Just enough. We add the diagonal, and then, and this is the delicate point, everything that the four operations allow us to manufacture from it and the old fractions: its multiples, its sums, what we obtain by dividing it, by combining it. Adding the diagonal alone would not be enough; the expanded world would immediately cease to be closed, since one could exit it at the very first calculation. We must therefore add, in a single gesture, the diagonal *and* all its descendants. We then obtain a slightly larger territory, which contains all of the old one, contains the diagonal, and remains closed: one can again add, subtract, multiply, and divide there without ever leaving. This new land, custom-made to accommodate the solutions of an equation, is what we will call its *land of roots*.

Each equation has its own. Not just any land: the smallest world where all its solutions finally become visible, where the equation completely breaks down into its simplest pieces, like a building returned to its stones. It is its minimal land, its natural territory.

And the size of this land climbs fast. For a second-degree equation, two dimensions are enough to navigate it. For the third degree, the land can demand six; for the fourth, twenty-four; for the fifth, up to one hundred and twenty. The territory swells like a tide.

But, and this is the heart of this entire book, it is not size that matters. It is even the hardest lesson to accept, because it goes against intuition: we always believe that a large land is a complicated land, and that a small land is a simple land. That is false. An immense territory can have a clear, regular, predictable plan. A small territory can hide a labyrinth. What decides the fate of an equation is not the extent of its land, it is its *shape*: the way its regions nest within one another, the interior arrangement, the plan of the house.

Two images help here, and we must keep them in mind. Counting the trees tells nothing about the forest: one can know their exact number without knowing anything of how it is laid out, the clearings, the paths, what repeats. And to understand a river, a geographer does not examine its drops one by one, which would be endless and useless: he studies the basin that gathers them, the relief that decides, in silence, where each drop will go. The solutions of an equation are the drops. The land of roots is the basin. It is the relief that must be read.

Galois was not the first to enlarge these worlds. Before him, many mathematicians had added a root to the land of fractions, as one adds a room to an already built house, to accommodate what overflowed. It was useful, and no one was surprised by it. But no one stepped back to look at the whole.

Galois looked at the entire house. No longer just the room that had just been added, but the complete plan: how many floors, how they communicate, by which stairs, according to what symmetry. It is a seemingly tiny shift, since one is looking at the same house, yet everything changes. The one who adds a room solves a specific problem; the one who reads the plan understands why the house can, or cannot, be built in this way.

This simple shift of the gaze did not yet solve anything. It must be said honestly, so as not to cheat the reader: at this stage, we have no formula, we still do not know how to say which equation yields and which resists, none of the questions left by Abel are decided. Nothing is resolved. But everything is prepared. The stage is finally set, the decor is in place, and the answer will, one day, be able to play out there. For, once the right world is built, a new question, which one could not even ask before, becomes possible: what rearrangements does this world tolerate without anything breaking? What symmetries does this land of roots hide?

This is exactly what a seventeen-year-old boy was reading between the lines of Lagrange, in a study hall at Louis-le-Grand.

4

The Invisible Symmetries

A seventeen-year-old boy was reading Lagrange in a study hall at Louis-le-Grand. It was in 1828 or 1829; the exact date is lost, but not the place. The college was a severe boarding school in the heart of the Latin Quarter, with grey walls, paved courtyards, and bells that carved up the hours. Évariste Galois was no longer listening to them. He had stopped attending most of his classes. His literature teachers called him distracted, and his mathematics teachers found him arrogant.

He was reading. Not a student textbook: the *Réflexions sur la résolution algébrique des équations*, a text that Lagrange had written in 1770 for specialists. It was not something to be put in the hands of a teenager. No one had recommended it to him. He had found it alone, and he was reading it alone.

He was not reading it as one prepares for an exam. He was reading it as an architect reads unfinished plans. Lagrange had spotted the right object and asked the right question. When one has an equation and its solutions, what happens if one exchanges these solutions with one another? Lagrange had begun to look. Then he had stopped. He had dug the foundations and left the construction site open.

Galois saw the construction site. And he saw what had to be built there.

It is difficult to picture. A boy who is not of legal age to enroll in university, poorly graded, unloved by his masters, failing exams he despises, and who, during this time, in a common study hall, is following a trail that the greatest mathematicians of Europe had left abandoned. He was not following the lessons. He was elsewhere, in a low-voiced conversation with a sixty-year-old text, over the heads of his teachers.

To understand what he saw, let us take the simplest example in the world: the diagonal of a square of side one. This number is not a fraction; to accommodate it, one must enlarge the world of fractions slightly, just enough. In this expanded world live two twin roots: the diagonal, and its exact opposite, its reflection on the other side of zero. They are inseparable. Neither arrives without the other.

Now, make a gesture. Wherever you see the diagonal, put its opposite. And wherever you see the opposite, put the diagonal. You have just exchanged the two roots.

What broke? Nothing.

This is what one must feel. Count all the additions, all the products that held true in this world before the exchange: they still hold after. The gesture has permuted the two roots, and yet no relation, no equality has budged. The world has been turned inside out like a glove, without the slightest tear in its frame. The ordinary fractions, for their part, have not budged by a hair: they had nothing to exchange.

One can represent the gesture as a mirror. Place each number of the expanded world somewhere on a large sheet of paper. The exchange acts like a reflection: it folds each point onto its image on the other side of a line. The fractions are right on the mirror line. They reflect onto themselves. They do not feel a thing. Everything else tips over, and everything remains coherent.

It is a symmetry. Not the symmetry of a figure, of a butterfly or a snowflake: the symmetry of a world of numbers.

How many such gestures are there for this simplest of worlds? Exactly two. The first changes nothing: one leaves everything in place. The second is the mirror. There is no third, and the reason is beautiful in its spareness. Such a gesture must send a root of the equation to a root of the same equation. But the equation has only two roots: the diagonal and its opposite. Two possible destinations, therefore two possible gestures. Not one more.

These gestures have another property, gentle and clean. Perform the mirror, then perform it again: you have returned to your starting point. Two successive reflections cancel the reflection. The mirror undoes itself. Doing nothing and reflecting, reflecting twice and doing nothing: these two gestures, chained in any direction, never go outside themselves. They form a small closed system, which folds perfectly back on itself.

This is the idea. The gestures that exchange roots without breaking anything are the *symmetries of the equation*. It is not the roots themselves that count: their value, their decimals, do not matter. What matters is the way they can be permuted without the edifice collapsing.

And these symmetries are *invisible*. This is the right word, and we must pause on it. One does not read them on the written equation. One does not guess them in its digits. The equation, as it is posed on paper, shows nothing of this interior dance. To see it, one must first build the right world, the minimal land where all the roots coexist, and then observe the ways of rearranging it without breaking it. The symmetries do not inhabit the equation. They inhabit the world it engenders. Without this world, they have no stage on which to appear.

Let us remember this image, because it governs everything that follows. An equation has two faces. The written face, the one we put on paper with its figures and its equals sign: mute, closed, it yields nothing of its symmetries. And the hidden face, which only appears in the land of roots: the play of exchanges that break nothing. All of Galois's genius lies in a shift of the gaze. He stopped staring at the written face, as had been done for four thousand years, to go and see the hidden face, where the secret of the equation is actually written. This shift, changing terrain to make visible what was not visible, will be the gesture that, until the end of this book, moves everything forward.

This is why algebraists missed them for so long. For centuries, they manipulated roots without ever asking what exchanges they tolerated. One does not look for what one cannot name. Lagrange, first, had begun to look in this direction. His intuition was good. But he had neither the right world nor the right word for what he would have seen. He had opened a door and had not entered. Galois entered.

For the diagonal of the square, the scene is minuscule: two gestures, a mirror and immobility. But imagine a richer equation, with three roots, four, five. The number of gestures swells quickly. And above all, the way they link together becomes subtle.

There, a strange thing can occur. Make a gesture, then another: you get a certain rearrangement of the roots. Make the same two gestures in the reverse order, the second first, the first second: you can end up with a different rearrangement. The order in which one chains symmetries begins to matter. This is not a caprice or a detail of calculation. It is pure information. This sensitivity to order says something profound about the equation, something that no other view reveals, and that the following chapters will learn to read. For the diagonal of the square, order did not matter: a single mirror, nothing to mix up. As soon as there is a crowd, order speaks.

Galois saw all of this from his grey study hall. But the world did not see it. At the beginning of 1829, sure of holding something immense, he drafted a first memoir and entrusted it to the Académie des sciences through the mediation of Augustin-Louis Cauchy, the greatest analyst in France. Cauchy promised to present it. He did not. The memoir disappeared, and Galois never saw it again. He would write others. Fourier, perpetual secretary of the Académie, died before reading the next one, which went astray with him. Poisson examined the third, judged it obscure, and sent it back.

But in that room, between Lagrange and these refusals, the essential had been seen. Gestures that break nothing, that compose with one another according to regular laws, forming a closed and coherent system. He needed a word to name this system of exchanges. He would not find it in any treatise. He would invent it himself.

5

The Invention of a Word

In his drafts, Galois wrote a word. He wrote "group."

In the sense he meant it, this word did not exist. It was in no mathematics dictionary, in no course, in no treatise. The precise definitions, the scholarly vocabulary, the theory in due and proper form: all of that would come after him, forged by mathematicians who would read his manuscripts half a century later. But the idea itself was whole. Galois had understood that the symmetries of an equation do not form a bag of gestures thrown together. They form a system. And this system can be described with almost nothing.

What, then, is a group? Let us forget all technique. A group is the *grammar* of a play of symmetries. Not the list of gestures: the way they chain together, undo themselves, fold back on themselves. A grammar does not say what words you will speak. It says how words combine to remain language. Similarly, the grammar of a play of symmetries says how gestures combine while remaining symmetries. Chain two of them together: you still get a gesture in the play. There exists the gesture that changes nothing. Each gesture has its inverse gesture, the one that undoes it exactly, just as the key that locks a padlock is the key that opens it, turned in reverse. The system never leaks outside of itself.

This grammar is extraordinarily economical. It says nothing about the nature of the gestures, nor their number, nor what they act upon. That is where its power comes from. The same grammar holds for things with no apparent connection. The ways of turning an equilateral triangle onto itself without changing its appearance: six gestures in all, three rotations and three reflections. The hours that one can make the hand of a dial point to: a game that turns and always comes back round to where it began. The ways of shuffling five cards: one hundred and twenty possible shuffles. Three entirely different plays, a single grammar. And this grammar has a measure: the size of the play. The more symmetries there are, the greater the effort it took to accommodate the roots of an equation. The size of the group measures this effort exactly.

Here now is the theorem that Galois glimpsed, and that subsequent generations finished writing. Stated clearly, without an ounce of technique, it says this. To each equation corresponds a play of symmetries. And the grammar of this play is the *faithful portrait* of the architecture of the land of roots. To each floor of the land, that is, each intermediate step between the starting fractions and the complete world where the roots live, corresponds a precise piece of the play of symmetries. To know one is to know the other. The portrait has neither blur nor blind spot.

Everything then plays out on a single mental gesture. Let us return to the image of the watchmaker. A watch is understood if one can dismantle it: remove the case, then the dial, then each gear, each screw, down to the last piece, each separating cleanly from the next. As long as the pieces detach, one holds the entire mechanism. But if the watchmaker hits a welded block, a piece whose parts do not separate, he stumbles. He can go no further.

Here is the heart of the whole story. Solving an equation by a formula, that is, by those finite paths that employ only the four operations from school and the extraction of roots, is exactly being able to dismantle its play of symmetries, floor by floor, down to the last piece. Each root extraction is a floor that detaches cleanly. If the play can be completely dismantled, the equation is solved, root after root. If it contains a welded block, the formula does not exist. Not “we have not found it”: it does not exist, and we know why.

The second degree, the third, the fourth: their plays of symmetries can be dismantled. For two objects, for three, for four, everything detaches piece by piece. The old second-degree formula, with its single square root, is a single floor that falls away. Cardano's formula for the third degree, with its square root nestled under a cube root, is a dismantling in two floors. Ferrari's formula for the fourth degree, with its layers of nested extractions, is several floors. Each formula inherited from the Renaissance is nothing other than the translation, into operations, of a successful dismantling.

And at the fifth degree?

There, something changes in nature. Among the one hundred and twenty ways of shuffling five objects hides a piece that refuses all dismantling. A welded block. For two, three, four objects, this block does not appear: everything separates. Starting from five, it is there, massive, without cracks, without a handhold. No gesture, no ruse can divide it into simpler floors, just as no skill can cut steps into a seamless rock.

We must measure the difference with what precedes. At the fourth degree, the dismantling succeeded, but barely, like a lock that yields under the last key on the ring, just before we give up. At the fifth, the lock no longer yields at all. The first floor still detaches cleanly; we breathe; we think it will pass. Then we hit the block, and the descent stops short, like a staircase whose steps vanish above the void. This block, specialists name differently and describe with their tools; but the idea holds entirely in the image of the watchmaker before a mechanism that no longer lets itself be opened.

Let us insist, for this is the entire book in embryo. This is not a lack of ingenuity. It is not that mathematicians before Galois were less clever. It is a fact of architecture, written in the very shape of the play of symmetries. And it is final: it holds not only for the fifth degree, but for the sixth, the seventh, the hundredth, and every degree beyond four. The wall never ends.

Galois's theorem then fits in a single sentence. An equation is solved by a formula if, and only if, its play of symmetries can be completely dismantled. No exception, no twisted case, no reservation. And the theorem says even more: it does not judge a degree in bulk, it judges each equation, one by one. Certain equations of the fifth degree yield: the one asking for the number whose fifth power is two is solved at once, its solution being a simple fifth root. Others resist. And we know how to *read*, in the shape of the play of symmetries, which is which. It is a diagnosis, sharp as a blade, where Abel could only deliver a blanket verdict.

This is what must be remembered, and what this book will hammer home to the end. Galois did not persist in attacking the equation directly. He did something else: he transported the question elsewhere. The question "does this equation have a formula?", insoluble on its own terrain for four thousand years, becomes "can this play of symmetries be dismantled?", a question we know how to answer. The problem has not changed. The terrain has changed. This is a strategist's maneuver: choosing one's battlefield to gain an advantage that the other terrain forbade. Every great advance in this history will be this same gesture, repeated from century to century: shifting the gaze, changing terrain, carrying the problem where it finally becomes readable.

On September 4, 1843, eleven years after Galois's death, Joseph Liouville stood up before the Académie des sciences in Paris. He was thirty-four years old, held a chair at the Collège de France, ran a journal, and possessed a rigor that put charlatans to flight. He held in his hand sheets that he had spent months deciphering: Galois's manuscripts, transmitted by a faithful friend, forgotten in a drawer, finally read. Liouville had redone the calculations. Filled in the gaps. Followed to the end reasonings that skipped steps that their author alone leaped in a single bound. And each time, the conclusion held. He declared before his colleagues that these memoirs were "as correct as they are profound." The word was measured, as Liouville was. Its reach was not. It was the first endorsement of weight given to a work already more than a decade old.

All of this, verified, complete, and offered to readers without the tools to read it. It remained to be seen what would become of the man.

6
Évariste

Évariste Galois was born in Bourg-la-Reine, south of Paris, on October 25, 1811. His father, Nicolas-Gabriel, was the mayor of the commune: a cultivated, liberal man who composed satirical verses for village festivals. His mother, Adélaïde-Marie Demante, the daughter of a jurist, gave him a solid classical education, focused on Latin and literature. Nothing in this provincial childhood gave any hint of what would follow. It was a peaceful, literate household, without anything exceptional. The exception would come from within, without warning.

At twelve, he entered the Lycée Louis-le-Grand in Paris. The first years were ordinary. Then, around fifteen, he opened Legendre's textbooks and Lagrange's memoirs. Something shifted. It is said that he read Legendre like one reads a novel, in one go, without looking back. The rest ceased to interest him. The teachers noted a distracted, arrogant, impatient student, who skipped steps and refused to justify what he judged obvious. He was right most of the time. He had neither the age, the rank, nor the patience to have it accepted.

He twice attempted the entrance exam for the École Polytechnique, the royal road for French mathematicians. He failed both times. Legend has it that on the second try, exasperated by questions he found trivial, he threw the blackboard eraser at the examiner's head. The anecdote is undoubtedly embellished. The temperament it depicts, however, is accurate. Galois could tolerate no one's mediocrity, and least of all in an exam room. He fell back on the École Préparatoire, the future École Normale Supérieure, which was less prestigious, and which he hated. He was expelled from it in January 1831 for publishing a stinging letter against the director in a newspaper.

In the midst of all this, a bereavement. On July 2, 1829, his father hanged himself. Nicolas-Gabriel Galois, a mayor harassed by calumnies that the priest of Bourg-la-Reine had orchestrated against him, had not held out. Évariste was seventeen. The suicide fell a few days before his second attempt at Polytechnique. Imagine: a son who learns of the death of his father, and who must, almost immediately, present himself before a jury. Pain, rage, and isolation mixed in him with the certainty of his genius and contempt for the institutions that refused to see him.

The France of 1830 was a cauldron. The July Revolution had overthrown Charles X, but installed Louis-Philippe on the throne, whom the republicans held to be a bourgeois usurper. Galois threw himself into it without reservation. He frequented the *Société des amis du peuple*, demonstrated, and was arrested twice. The first time, in June 1831, for illegal possession of weapons during a republican banquet: acquitted. The second time, in July, for the same reason: six months in prison at Sainte-Pélagie. He spent the autumn and winter there, working on his mathematics in a cell shared with other political detainees. Genius continued to write between the bars, as if it had not noticed the bars.

He came out weakened, feverish. Cholera was ravaging Paris that year; he had contracted it. He was transferred to a nursing home, and then freed for good on April 29, 1832. The following weeks are poorly illuminated. He lodged near the Observatory, frequented republican circles again, and worked on his memoirs with an intensity that worried those close to him. Then came the matter of the duel. Two centuries of research have not untangled its causes: a matter of honor, perhaps linked to a young woman, Stéphanie Poterin du Motel, perhaps to rivalries within the movement. Historians get lost in it. Only one thing is certain. The night before, Galois knew he was going to fight.

Let us return now to that night. You know it already: a room,