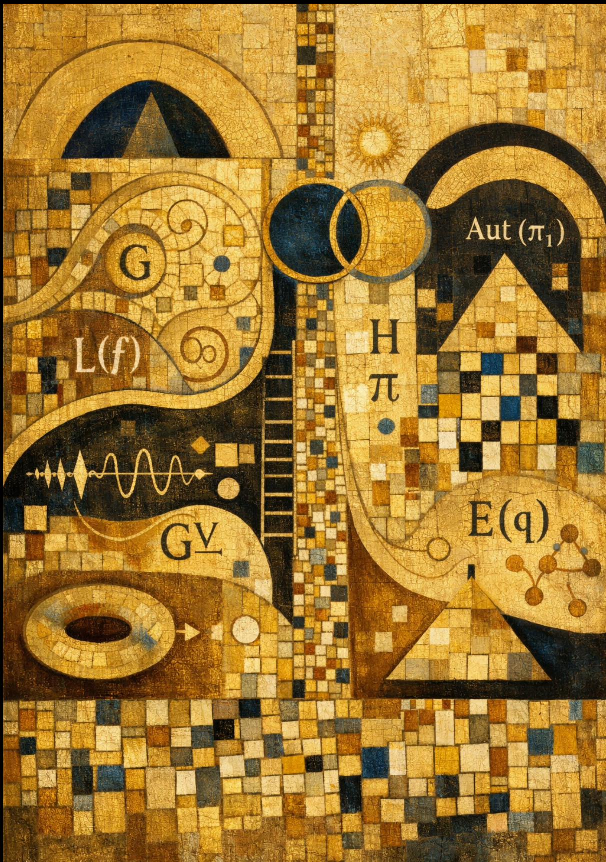
The Apocalypse According to Galois



The Revelation of Hidden Symmetries

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The Revelation of Hidden Symmetries

William de France

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*"Mathematics is the art of giving
the same name to different things."*

Henri Poincaré

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Prologue

The candle was down to an inch of wax. The young man moved it to the edge of the table, away from the papers, and picked up his pen. He was twenty years old. In a few hours he would fight a duel. He had known since the day before.

What he was writing was not a will. It was not a love letter, not a political manifesto. It was mathematics. Notes in the margins of memoirs that no one had read, corrections scrawled in haste, fragments of a theory of which he was the sole guardian. He was writing to his friend Chevalier, the only person to whom he could entrust what mattered. In the margin of a draft, he scribbled: ”I have not the time.”

What he saw, that night, were the invisible symmetries of equations. He saw that to each equation there corresponds a set of transformations, and that the structure of that set decides whether the equation can be solved by a formula or whether it resists forever. He saw that impossibility is not a blank wall but the sign of an architecture. And he saw, further still, that this architecture extends far beyond equations, toward territories whose contours he could only glimpse.

Évariste Galois died the following day, May 31, 1832. Fourteen years later, Joseph Liouville published his manuscripts. The world took a century to unfold them.

The Greek word *apokalypsis* does not mean catastrophe. It means unveiling: the act of removing a veil from something that was present but invisible. This book tells the story of such an unveiling. It begins long before Galois, with the scribes of Mesopotamia who carved recipes for calculation on clay tablets. It does not end with him. The idea that Galois glimpsed in too short a night grew far beyond anything he could have imagined. It spread through physics, geometry, number theory. A century later, it gave rise to the Langlands program, a dictionary between mathematical worlds that everything seemed to separate. And in 2024, a proof showed that this dictionary, in its geometric version, is an identity of structures.

Three unveilings. Three veils lifted. The story that follows tells them in the order in which they appeared.

Act One

Impossibility

1

The scribes and the formula

Around 2000 BCE, in a Mesopotamian city whose name is lost, a scribe pressed his stylus into a tablet of fresh clay. He was writing a recipe. Not a recipe for food, not a religious ritual, but a procedure for calculation: given a square whose area is known, find the length of the side. The scribe possessed no algebraic notation. He did not speak of unknowns or equations. He stated the problem in words, Akkadian or Sumerian words whose precise translation still divides Assyriologists. But the method he inscribed on the clay, a sequence of arithmetic operations applied to the data of the problem, is essentially the one that schoolchildren around the world still learn today. The Babylonian scribes were solving the second degree.

They did not know it. The very idea of a second-degree equation presupposes an abstraction that did not yet exist. The scribes worked on concrete problems: fields to measure, volumes to calculate, shares of inheritance to divide. One such problem, recorded on the tablet known today as YBC 6967, held at Yale University, asks this: find two numbers whose product is 60 and whose difference is 7. The scribe gives the answer step by step, dividing, squaring, extracting a root, adding. The procedure is what we would recognize as the solution of $x^2 - 7x - 60 = 0$, but the scribe would never have written it that way. Each problem received its own recipe, and these recipes were recorded on tablets that served as teaching manuals in the scribal schools. Hundreds of such tablets have survived. They show an impressive mastery of numerical calculation, a keen sense of regularity, and, in places, what looks like an intuition of the underlying structure. But between the scribe's recipe and the mathematician's formula, there is a conceptual gulf that only centuries would bridge.

That gulf is the notion of *solvability by radicals*. To measure it, one must understand what mathematicians mean when they say an equation is "solved." The word does not simply mean that a number has been found that works. It does not mean that an approximation has been obtained, however precise, or that an algorithm converges to the right value after infinitely many steps. To solve an equation, in the strict sense of classical algebra, is to write its roots in explicit form: a finite expression, built from the coefficients of the equation using only five operations. The first four are the ordinary operations of arithmetic: addition, subtraction, multiplication, and division. They do not lead very far. If the coefficients are rational numbers, ordinary fractions, the four operations produce only more rational numbers. One stays in the same world. It is the fifth operation, the extraction of roots, that allows one to cross the boundary. Extracting a square root, a cube root, an nth root, is to manufacture numbers that did not exist in the starting world. It is the tool that gives the formula its power.

The quadratic formula is the most familiar case. Given the equation $ax^2 + bx + c = 0$, one computes the discriminant $b^2 - 4ac$, extracts its square root, and the two solutions follow immediately. The formula fits on a single line. It works regardless of the coefficients. It is universal in the strongest sense: it does not depend on the particular values of a, b, and c. Whether the coefficients are small or large, positive or negative, whole or fractional, the same recipe applies and produces the answer. Behind its apparent simplicity lies a remarkable fact: a single extraction of a root suffices. One enters the calculation with rational numbers, performs a few additions and multiplications, extracts a square root, and emerges with the answer. The operation is clean, definitive, without remainder. The Babylonian scribe, with his words and his tablets, was doing exactly this, without the vocabulary to say so.

The power of this formula is worth pausing over, for it foreshadows everything that follows. To solve an equation by radicals is to build a path that starts from the coefficients and arrives at the roots using only the five operations. The path is finite. It has a beginning and an end. The number of steps is determined in advance. And the last step, the one that extracts the root, is also the one that creates the genuinely new number, the one that did not exist in the world of the coefficients. Without the extraction of a root, one never leaves the territory of the rationals. With it, one reaches irrational numbers, numbers the Greeks already knew, like $\sqrt{2}$, but whose deep nature would not be understood until much later.

The vocabulary needed to say all this took mathematicians centuries to forge. The quadratic formula had been known, in one form or another, since antiquity. The Greeks reformulated it in geometric terms, constructing roots with straightedge and compass. Euclid, in the *Elements*, proposed constructions that amount to solving second-degree equations, but in the language of lengths and areas, without ever writing an algebraic symbol. The Arab mathematicians took a decisive step. Al-Khwarizmi, in ninth-century Baghdad, wrote a treatise whose title, *al-Kitab al-mukhtasar fi hisab al-jabr wa'l-muqabala*, gave its name to algebra. He classified second-degree equations into six types and gave a systematic method of solution for each. For the first time, solving equations became an organized art, with rules and procedures. But throughout this period, the second degree remained the summit of the art. Equations of higher degree resisted. Special cases could be handled, numerical methods could approximate the roots, but the fundamental question remained open: is there a formula for the third degree? For the fourth? For the fifth?

For more than three thousand years, the second degree remained the frontier. The Indians, Brahmagupta in the seventh century, Bhaskara in the twelfth, refined the methods. The medieval European mathematicians, trained in the Arab tradition, transmitted and adapted them. But no one crossed the threshold of the third degree. The question hung suspended, like an unkept promise: if the second yields to a formula, the third must yield too. The confidence was intact. Algebra had always found the formula in the end. One simply had to search long enough, with enough ingenuity.

We shall soon see how the Italian Renaissance forced open the gates of the third and fourth degree, in circumstances that owed as much to intellectual dueling as to patient research. But before telling that story, it is worth retaining the lesson that the second degree already teaches. To solve an equation by radicals is to bring the coefficients down through a sequence of elementary operations, of which the last, the extraction of a root, is the only one that truly creates something new. The quadratic formula requires only a single extraction. Those of the third and fourth degree will require more, nested inside one another with mounting complexity. And this growth in complexity will carry within it a warning that no one, for a long time, will know how to read.

The clay tablets survived four thousand years. The formulas they contain still work. But the question they posed in silence, the question of how far the vocabulary of five operations can reach, would wait nearly four millennia before receiving its answer. Three thousand five hundred years separate the scribes of Mesopotamia from the algebraists of the Renaissance. Then, in sixteenth-century Italy, the race begins again.

2

The Renaissance race

In Bologna, in February 1535, a crowd gathered in a hall where two mathematicians were about to face off. The challenge was public, as was the custom in Renaissance Italy: each contestant submitted to the other a list of thirty problems, and the one who solved the most within the allotted time won the contest, a banquet, and the reputation that came with it. On one side stood Antonio Maria Fior, a student of the late Scipione del Ferro, armed with a secret method that his master had passed on from his deathbed. On the other stood Niccolò Fontana, nicknamed Tartaglia, the stammerer, a self-taught mathematician from Brescia who had learned mathematics on his own after French soldiers slashed his face in childhood. Fior possessed the formula for a particular type of cubic equation. He believed this would suffice. He was wrong.

Tartaglia solved all thirty of Fior's problems in two hours. Fior solved none of his. The victory was total. Tartaglia had found, in the days before the contest, a general method for cubics, and he had kept it to himself. The secret was worth gold in the literal sense: university chairs were won in these public jousts, and the formula was the decisive weapon. Tartaglia had no intention of sharing it.

He shared it anyway. Gerolamo Cardano, physician, mathematician, astrologer, gambler, one of the most brilliant and most troubling minds of his century, persuaded Tartaglia to confide it under a pledge of secrecy. Tartaglia yielded, reluctantly, in 1539. Cardano swore to publish nothing. Then he published. In 1545, in his *Ars Magna*, he set out the complete method for solving the third degree, attributing the initial discovery to del Ferro and the generalization to Tartaglia. Tartaglia cried foul. The quarrel poisoned the last twelve years of his life.

The affair was human, petty, full of resentment and wounded vanity. But the formula itself was magnificent. Magnificent and formidable. To solve the reduced equation $x^3 + px + q = 0$, Cardano's formula reads:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Two cube roots nested inside a square root. The expression is already notably complex. But the most troubling thing is not its length. It is that the formula conceals a trap: in certain cases, when the equation has three distinct real roots, the formula produces square roots of negative numbers. The equation has perfectly real solutions, traceable on a graph, but the formula passes through quantities that are not real numbers. They would later be called *imaginary* numbers. Cardano himself noted the phenomenon with perplexity, calling it "subtle to the point of uselessness." But the formula worked. By combining these strange quantities according to the rules of calculation, one always landed back on the right roots. The imaginary numbers were not a flaw in the method. They were the price to pay for descending into the depths of the equation and resurfacing with the solution.

The story of the cubic formula is the story of a secret that could not be kept. Del Ferro had found the method first, around 1515, in his office at the University of Bologna. He published nothing. He confided it to his student Fior, who lost it in the duel against Tartaglia. Tartaglia confided it to Cardano, who published it. Three men, three betrayals of trust, and at the end of the road a formula that now belonged to the world. Mathematics sometimes advances not through the generosity of its discoverers, but despite their jealousy.

The fourth degree fell almost immediately. Lodovico Ferrari, Cardano's student, barely eighteen years old, found a method in 1540. The idea was to reduce the fourth-degree equation to an auxiliary equation of the third degree, which one now knew how to solve. The resulting formula was monstrous. Three nested root extractions, expressions filling an entire page, case after case to consider. But the principle held: one combined the five elementary operations, and one reached the roots. Ferrari had proved that the fourth degree was solvable by radicals. Cardano published this result, too, in the *Ars Magna*.

The third degree in a decade, the fourth almost in its wake. The progression seemed irresistible. In three quarters of a century, Italian algebra had conquered two fortresses that three thousand years of mathematics had left intact. The confidence was total: the fifth degree would yield in its turn. One only needed to find the right substitution, the right trick, the right angle of attack.

But something troubling was showing through this triumphal progression. The quadratic formula fits on a single line. The cubic fills a paragraph. The quartic takes a page. At each degree, the complexity did not add up: it multiplied. The roots nested inside one another like Russian dolls, each layer adding another extraction. The second degree required only one square root. The third added two cube roots. The fourth piled everything together. This was not a simple lengthening. It was a change of nature. The difficulty grew not arithmetically but structurally, as though each additional degree added not one more stair to a staircase, but an entire floor to a building whose architecture grew more intricate at every level.

No one knew how to read this warning. For nearly two centuries, the finest minds in Europe searched for the formula of the fifth degree. Euler, the most prolific of all mathematicians, tried and failed. Lagrange, in the late eighteenth century, undertook a systematic study of the question with unprecedented depth. He examined the methods that had worked for degrees 2, 3, and 4, sought to understand what made them possible, and discovered that all of them rested on a common idea: certain combinations of the roots, called *resolvents*, possess symmetry properties that make them accessible to calculation. For the lower degrees, these resolvents always had a lower degree than the original equation, which allowed one to solve them first and then work back toward the roots. But for degree 5, Lagrange's resolvent was of degree 6. Instead of simplifying the problem, it made it harder. Lagrange noted this fact with remarkable lucidity, but he did not draw the ultimate conclusion. He had laid the foundations of an edifice whose summit he could not yet see.

Tschirnhaus, Bézout, Bring, Jerrard, so many others pushed the calculations in every direction. They tried ingenious substitutions, reductions of form, methods of descent. No one found the formula. The suspicion grew slowly: perhaps the formula did not exist. Not that it was simply too complicated to write down, but that it was, in a rigorous sense, impossible. The idea was dizzying. The habit of centuries had inscribed in the minds of mathematicians a tacit equivalence between ”equation” and ”formula.” Every equation, they believed, must possess one; it was only a matter of finding it. To challenge this equivalence was to rethink the very nature of what it means to solve. How does one prove that a thing does not exist? How does one show, not that one has failed to find it, but that there is nothing to find? Such an argument would no longer be a matter of calculation. It would be a matter of structure.

The third degree had fallen in the clamor of the Bolognese duels. The fourth had yielded in its wake. The fifth remained. Two centuries of searching would change nothing.

You have just read the prologue and the first two chapters.

The complete book contains 24 chapters in three acts:

Act One — Impossibility

The scribes and the formula · The Renaissance race · The impossible · The world of roots · Invisible symmetries · The group · Solvable and irreducible · The theorem · The night and the duel

Act Two — The unfolding

Jordan's century · Symmetries of the physical world · Deep arithmetic · Representations · First unveiling

Act Three — The correspondence

The letter · Two worlds · The number 691 · Fermat falls · Local, global, and the trace formula · The fundamental lemma · Geometric Langlands · The 2024 proof · The meeting with physics · The veil lifted

Plus four appendices: glossary, timeline, gallery of key figures, annotated bibliography.