



Articulated Course
in
Mathematics
V
Mathematical Physics

FROM SYMMETRIES TO STRINGS

William de France

CLEMENTINIUM EDITIONS

Clem

MATHEMATICAL PHYSICS

Mathematics is the language in which God has written the universe.

GALILEO (EDITORIAL TRANSLATION OF THE FRENCH EPIGRAPH)

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PREFACE

This fifth volume studies how mathematical structures become physical descriptions. A symmetry specifies possible transformations; it does not choose a dynamics by itself. An action yields equations; it does not yet determine their quantization. An effective theory produces predictions at a specified precision; it does not claim to describe every scale.

The route follows these distinctions. Mechanics, representations, and relativity prepare quantum states and collective systems. Field geometry leads to quantization, changes of scale, and gauge theories. Exact sectors and dualities then open onto the construction of strings and holographic dictionaries. A descending route finally recovers certain physical regimes through calculated limits, before confronting the constructions with their free parameters and other approaches to quantum gravity.

The forty-four chapters bring together four hundred and six articles. Each develops an idea through a calculation whose assumptions and scope remain visible. Further readings lead back to the sources. A solved research question extends each chapter: it tests a construction or distinguishes two operations; it does not claim to solve a general open problem.

Volume IV leads to geometric Langlands and the course of its proof in the setting studied. Here its physical interpretation returns through gauge dualities and branes, without being confused with that proof. The two volumes speak to each other; either can be approached using the tools of Volume III.

W. de F.

TO THE READER

This course articulates, in five volumes, the grammar through which reality composes itself: mathematical articles from the first integer to the Langlands correspondence, and from the brachistochrone to the amplituhedron. The undertaking is vast. The author does not claim it is flawless; he affirms that it is sincere.

Each article develops one idea, from motivation to interpretation. Definitions are framed; major theorems are also framed, in red, in Volumes 2 and 3 of the collection. Proofs are given where they illuminate, sketched where full rigor would obscure the argument, and omitted explicitly when their difficulty exceeds the article's scope. The reader will be able to distinguish these three cases.

Figures accompany the calculations; they show the objects and regimes specified in the text. The same structure reappears in several forms along the route. Its return lets us gauge what a new description preserves and what it adds.

Each chapter continues with Further reading, chosen to reach the sources and deepen the constructions, then with Research questions. These invite readers to extend an example, specify an assumption, or compare two operations. When a conjecture enters, its status is distinguished from proved results; not all these questions are open problems in current research.

The reader may follow the linear route or take passages across volumes. Volumes I–III form a continuous progression from secondary-level mathematics to advanced graduate material. Volume IV rises toward the Langlands program; Volume V toward mathematical physics. Both presuppose Volume III but can be read independently of each other.

In the English edition, verified English readings are used where appropriate. Original sources and attributions remain; section references match the editions cited. References retained in another language are identified as such.

Every reported error is an error to be corrected.

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NOTE ON THE LEVELS

The five volumes follow a progression from the first mathematical objects to questions of research:

- Elements (I): *secondary-school mathematics, from arithmetic to the first proofs.*
- Foundations (II): *the foundations of university mathematics.*
- Graduate Mathematics (III): *deeper study and preparation for research.*
- Correspondences (IV): *the research level, from class field theory to the Langlands program.*
- Mathematical Physics (V): *the research level, from symmetries to strings.*

These levels guide the reader without imposing an academic timetable. Parts and chapters mark the stages of the journey; the pace depends on the reader's background and the questions they wish to explore.

Volumes I, II, and III successively build the prerequisites for the next volume. Readers may enter at the level for which they have the necessary background. Volumes IV and V share the foundation of Graduate Mathematics and may be read independently of each other. The bridges at the beginning of each volume explain the change in perspective.

NOTATION

Local conventions are specified in the articles; the same letter may play different roles in different sectors. Electromagnetic couplings below use rationalized units.

$\hbar$	<i>reduced Planck constant $\hbar/2\pi$</i>
c	<i>speed of light in vacuum</i>
G	<i>Newton's gravitational constant (or a Lie group, depending on context)</i>
k_B	<i>Boltzmann constant</i>
e	<i>elementary charge</i>
α	<i>fine-structure constant $e^2/(4\pi\hbar c) \approx 1/137$</i>
ℓ_P	<i>Planck length $\sqrt{\hbar G/c^3}$</i>
$q, \dot{q}$	<i>generalized coordinates, generalized velocities</i>
p	<i>conjugate momentum</i>
$\mathcal{L}(q, \dot{q}, t)$	<i>Lagrangian</i>
$H(q, p, t)$	<i>Hamiltonian</i>
$S[q]$	<i>action functional</i>
$\{f, g\}$	<i>Poisson bracket</i>
ω	<i>symplectic form</i>
$\mu : M \rightarrow \mathfrak{g}^*$	<i>moment map</i>
$\mathcal{H}$	<i>Hilbert space (or Hamiltonian, depending on context)</i>
$ \psi\rangle, \langle\phi $	<i>ket, bra (Dirac notation)</i>
$\langle\phi \psi\rangle$	<i>inner product</i>
$\hat{A}$	<i>operator associated with the observable A</i>
X, P	<i>position and momentum operators</i>
$[A, B]$	<i>commutator $AB - BA$</i>
ΔA	<i>standard deviation (uncertainty) of the observable A</i>
σ_i	<i>Pauli matrices ($i = 1, 2, 3$)</i>
$a, a^\dagger$	<i>annihilation and creation operators</i>
$ n\rangle$	<i>occupation-number eigenstate</i>
ρ	<i>density matrix (state operator)</i>
$S(\theta)$	<i>scattering matrix</i>
$U(t) = e^{-iHt/\hbar}$	<i>evolution operator</i>
GL_n, SL_n	<i>general linear and special linear groups</i>

$O(n), SO(n)$	<i>orthogonal and special orthogonal groups</i>
$U(n), SU(n)$	<i>unitary and special unitary groups</i>
$Sp(2n)$	<i>symplectic group</i>
$SO(3, 1)$	<i>Lorentz group</i>
$SL(2, \mathbb{C})$	<i>double cover of $SO^+(3, 1)$</i>
$\mathcal{P}$	<i>Poincaré group = $SO^+(3, 1) \ltimes \mathbb{R}^4$</i>
$\mathfrak{g}$	<i>Lie algebra of a group G</i>
$\mathfrak{h}$	<i>Cartan subalgebra</i>
$\mathfrak{sl}_n, \mathfrak{su}_n, \mathfrak{so}_n$	<i>classical Lie algebras</i>
$\exp : \mathfrak{g} \rightarrow G$	<i>exponential map</i>
ad_X	<i>adjoint representation, $Y \mapsto [X, Y]$</i>
$(X Y), g_{ab}$	<i>Killing form</i>
W	<i>Weyl group</i>
Φ, Δ	<i>root system, simple roots</i>
$U(\mathfrak{g})$	<i>universal enveloping algebra</i>
$U_q(\mathfrak{g})$	<i>quantum group (Drinfeld–Jimbo deformation)</i>
$\rho : G \rightarrow GL(V)$	<i>group representation</i>
χ_ρ	<i>character of the representation ρ</i>
j	<i>spin (half-integer indexing the irreducibles of $SU(2)$)</i>
(j_1, j_2)	<i>finite-dimensional irreducible complex representations of $SL(2, \mathbb{C})$</i>
λ	<i>dominant weight (highest weight)</i>
$\hat{G}$	<i>complex dual group; ${}^L G$ denotes the L-group, with its Galois action</i>
$\pi : E \rightarrow M$	<i>vector or principal bundle</i>
$P \rightarrow M$	<i>principal G-bundle</i>
$T_p M, T_p^* M$	<i>tangent space, cotangent space at p</i>
$\Omega^k(M)$	<i>differential k-forms on M</i>
∇	<i>connection (covariant derivative)</i>
A	<i>connection potential (gauge field)</i>
F_A	<i>curvature of a connection A</i>
D_μ	<i>gauge-covariant derivative $\partial_\mu + gA_\mu$</i>
$F_{\mu\nu}$	<i>field-strength tensor</i>
c_1, c_n	<i>Chern classes</i>
$\hat{A}(M)$	<i>$\hat{A}$-genus (Atiyah–Singer)</i>
$\text{ind}(D)$	<i>index of an elliptic operator</i>
$YM(A)$	<i>Yang–Mills functional $\int F_A ^2$</i>
$CS(A)$	<i>Chern–Simons functional</i>
g_{YM}	<i>gauge coupling</i>
α_s	<i>strong coupling $g_{QCD}^2/(4\pi)$</i>
$\beta(g)$	<i>renormalization-group beta function</i>

ϕ, ψ, A_μ	<i>scalar, spinor, gauge fields</i>
$\mathcal{L}(\phi, \partial_\mu \phi)$	<i>Lagrangian density</i>
$T_{\mu\nu}$	<i>energy-momentum tensor</i>
$\mathcal{Z}$	<i>partition function (path integral)</i>
$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$	<i>n-point correlation function</i>
Λ	<i>ultraviolet cutoff (or cosmological constant)</i>
c	<i>central charge (Virasoro algebra)</i>
$L_n, \bar{L}_n$	<i>Virasoro modes</i>
$h, \bar{h}$	<i>conformal weights; $\Delta = h + \bar{h}$ in two dimensions</i>
$\mathcal{V}$	<i>vertex algebra</i>
$g_s = e^{\Phi_0}$	<i>string coupling fixed by the background dilaton</i>
α'	<i>Regge slope (tension $T = 1/(2\pi\alpha')$)</i>
$\ell_s = \sqrt{\alpha'}$	<i>string length</i>
$\tau = C_0 + ie^{-\Phi}$	<i>axiodilaton (type IIB)</i>
$X^\mu(\sigma, \tau)$	<i>string coordinates in target space</i>
d	<i>critical dimension ($d = 26$ bosonic, $d = 10$ superstring)</i>
$\mathcal{N}$	<i>number of supersymmetries</i>
$Q_\alpha, \bar{Q}_{\dot{\alpha}}$	<i>supercharges</i>
θ^α	<i>Grassmann variables</i>
$W(\Phi)$	<i>superpotential</i>
$g_{\mu\nu}$	<i>spacetime metric</i>
$\eta_{\mu\nu}$	<i>Minkowski metric $\text{diag}(-1, 1, 1, 1)$</i>
$\Gamma_{\mu\nu}^\lambda$	<i>Christoffel symbols</i>
$R^\rho{}_{\sigma\mu\nu}$	<i>Riemann tensor</i>
$R_{\mu\nu}, R$	<i>Ricci tensor, scalar curvature</i>
$G_{\mu\nu}$	<i>Einstein tensor $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$</i>
$r_s = 2GM/c^2$	<i>Schwarzschild radius</i>
G_{SM}	<i>gauge group $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$</i>
θ_W	<i>Weinberg angle</i>
$V_{\text{CKM}}, U_{\text{PMNS}}$	<i>mixing matrices (quarks, neutrinos)</i>
$v \approx 246 \text{ GeV}$	<i>Higgs field vacuum expectation value</i>
$\beta = 1/(k_B T)$	<i>inverse temperature</i>
$Z = \sum_n e^{-\beta E_n}$	<i>canonical partition function</i>
$M(T)$	<i>magnetization (order parameter)</i>
T_c	<i>critical temperature</i>
σ_{xy}	<i>Hall conductance</i>
$ 0\rangle, 1\rangle$	<i>computational basis of a qubit</i>
$S(\rho)$	<i>von Neumann entropy $-\text{Tr}(\rho \log \rho)$</i>
S_A	<i>entanglement entropy of region A</i>
$\mathcal{M}_H$	<i>Hitchin moduli space</i>

$\mathrm{Bun}_G(X)$	<i>stack of G-bundles on X</i>
$\mathcal{M}_{g,n}$	<i>moduli space of genus-g curves with n punctures</i>
Tr, tr	<i>trace (operator or representation)</i>
$H^k(M)$	<i>degree-k de Rham cohomology</i>
$\pi_1(X)$	<i>fundamental group of X</i>
$\chi(M)$	<i>Euler characteristic of M</i>
$\oplus, \otimes, \wedge$	<i>direct sum, tensor product, exterior product</i>
$\simeq, \cong$	<i>isomorphic to, homeomorphic/diffeomorphic to</i>

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FROM CORRESPONDENCES TO PHYSICAL REGIMES

The preceding volume studied correspondences: which objects to relate, which operations to preserve, and which assumptions make the identification possible. Physics takes up these questions, adding states, evolution, and observables. Two descriptions are compared not only through their spaces but through what they allow us to calculate in the same regime.

Mechanics offers a first example. A trajectory may be determined by a differential equation or by stationarity of an action. Passing between these formulations requires boundary conditions; passing to Hamilton also requires an appropriate Legendre transformation. Symmetries can then become conserved quantities. The structures are not simply juxtaposed: every link has conditions.

This requirement accompanies the entire volume. A quantum operator has a domain; a gauge field has redundancies; an effective theory has a scale of validity. Dualities, compactifications, and holographic dictionaries multiply the possible passages without removing the need to choose a state or control an approximation.

The reader will meet groups, bundles, cohomology, and categories again. The question now is which physical data they organize and what they leave undetermined. The first step is modest and decisive: varying a trajectory while knowing what is held fixed.

Part I

Dynamics, Symmetries, and Spacetime

Mechanics begins with a trajectory and the data that determine it. The variational principle leads to Lagrangian and Hamiltonian formulations; symmetries specify what is conserved. Lie groups and representations then supply the language of transformations. Causal geometry and Einstein's equations finally change the role of spacetime: the setting for dynamics itself becomes a dynamical object.

CHAPTER I

LAGRANGIAN MECHANICS

Recognizing a symmetry does not suffice to know how a system evolves. We must also choose a law, say what may vary, and specify the conditions under which motions are compared. Lagrangian mechanics gathers these data into an action. Its ambition lies in this change of perspective: a quantity attached to the entire path recovers an equation valid at every instant. The comparison erases neither forces nor constraints; it makes their effects calculable in suitable coordinates. At this first threshold of physics, correspondences between structures give way to a more precise question: what dynamics do the chosen data determine?

Articles in this chapter:

- 001.** *Principle of Stationary Action — Comparing paths does not promise a minimum.*
- 002.** *Euler–Lagrange Equations — A local variation imposes an equation of motion.*
- 003.** *Symmetries and Noether’s Theorem — The charge retains the contribution of the boundary term.*
- 004.** *Lagrangians with Constraints — The reaction completes the description of allowed motion.*
- 005.** *Stationarity, Minima, and the Second Variation — Duration changes the nature of the critical point.*
- 006.** *Brachistochrone and Catenary: One Identity — The chosen criterion determines the optimal curve.*

I. PRINCIPLE OF STATIONARY ACTION. *How* can a quantity computed over an entire path express a law of motion valid at every instant? A force describes what happens locally. The action assigns a number to a trajectory, then compares it with neighboring paths. This shift of perspective resembles a grammar that examines a whole sentence; the image gives the body neither an intention nor knowledge of its destination. It announces another formulation of its dynamics.

DEFINITION 1.1: Let Q be a configuration space, initially an open subset of $\mathbb{R}^n$. A *Lagrangian* is a smooth function $L(q, v, t)$. To a curve q on a fixed interval $[t_0, t_1]$, we associate the *action*

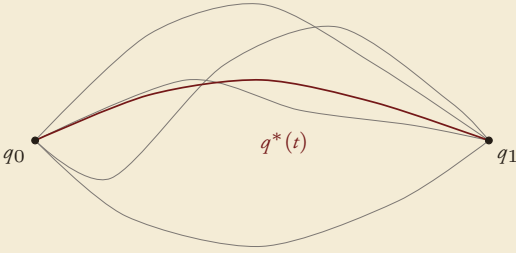
$$S[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t) dt.$$

An admissible variation q_ε preserves both endpoint positions. Its infinitesimal displacement $\eta = \partial_\varepsilon q_\varepsilon|_0$ therefore vanishes at t_0, t_1 . The *first variation* is

$$\delta S[q] \cdot \eta = \left. \frac{d}{d\varepsilon} S[q_\varepsilon] \right|_0.$$

The action is *stationary* if this derivative vanishes for every admissible variation.

On a manifold, L is defined on $TQ \times \mathbb{R}$ and $\eta(t)$ belongs to $T_{q(t)}Q$: the family of curves defines the variation, even when no single chart covers the path. In the calculations below, paths are of class C^2 .



$$\delta S[q^*] = 0$$

A family of paths with the same endpoints. The distinguished path is stationary for the model under consideration; the drawing does not depict the values of the action.

Hamilton's principle formulates the model's dynamics as $\delta S = 0$. The choice of L contains a physical assumption. For particles subject to a velocity-independent potential, the natural model $L = T - V$ will recover Newton's equations; not every Lagrangian has this form. The figure depicts comparisons between paths, not a selection made in anticipation. Specifying two positions at two times guarantees neither existence nor uniqueness of a trajectory: this problem differs from prescribing an initial position and velocity.

A MINIMUM WE CAN ESTABLISH: For a free particle on a line, $L = m\dot{q}^2/2$, with $m > 0$. The affine path

$$q_*(t) = q_0 + \frac{q_1 - q_0}{\Delta t}(t - t_0), \quad \Delta t = t_1 - t_0 > 0,$$

joins the endpoints at constant velocity. Every competitor has the form $q_* + \eta$, with $\eta(t_0) = \eta(t_1) = 0$. Expanding the square gives the cross term $m\dot{q}_* \int \dot{\eta} dt = 0$. Hence

$$S[q_* + \eta] - S[q_*] = \frac{m}{2} \int_{t_0}^{t_1} \dot{\eta}^2 dt \geq 0.$$

Equality requires $\dot{\eta} = 0$, then $\eta = 0$ by the boundary conditions: here the minimum is unique. This result follows from the nonnegative square, not from the word “action” alone.

The historical expression “least action” thus covers two questions: the vanishing of the first variation and the actual comparison of values. The oscillator will separate them in Article 5. For now, we still need to understand how the vanishing of a variation over the whole path imposes an equation at each instant.

2. EULER–LAGRANGE EQUATIONS. *AN* integral can vanish through cancellation between regions where its integrand has opposite signs. Stationarity requires more: the vanishing must withstand every variation of the path, including those affecting only a small interval. A requirement on the whole fabric can thus be tested thread by thread; the image becomes precise through the freedom to localize a variation.

THEOREM 2.1 (EULER–LAGRANGE): Let L be of class C^2 and q a C^2 curve in an open subset of $\mathbb{R}^n$, with fixed endpoints. The action is stationary at q if and only if

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n.$$

All derivatives of L are evaluated at $(q(t), \dot{q}(t), t)$.

PROOF: In the following formulas, repeated indices are summed. Set $p_i = \partial L / \partial \dot{q}^i$. Differentiating under the integral and integrating by parts gives

$$\begin{aligned} \delta S[q] \cdot \eta &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} \eta^i + p_i \dot{\eta}^i \right) dt \\ &= [p_i \eta^i]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \dot{p}_i \right) \eta^i dt. \end{aligned}$$

The boundary term vanishes. If a continuous coefficient of η^i were strictly positive at an interior point, it would remain so in a

neighborhood; a positive test function supported there would give a positive integral. The same argument handles a negative coefficient. This is the fundamental lemma of the calculus of variations: every coefficient vanishes. Conversely, the equations and boundary conditions make both terms in the last line vanish.

This equivalence gives a local condition for stationarity, without implying a minimum. If an endpoint were allowed to vary, the boundary term would remain and impose an additional condition suited to the problem; fixing endpoints is part of the data, not a computational formality.

NEWTON IN OTHER COORDINATES: For $L = m|\dot{q}|^2/2 - V(q)$, the equations become $m\ddot{q} = -\nabla V$. The variational formulation therefore recovers the chosen Newtonian model. It survives a smooth invertible change of coordinates because neither the paths nor their action values change. Covariance also transforms velocities; it does not mean substituting positions alone.

In the plane with the origin removed, polar coordinates give, for the free particle,

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2), \quad r > 0.$$

The two equations are

$$\ddot{r} = r\dot{\theta}^2, \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0.$$

The radial term expresses the changing polar directions; it adds no interaction to the free particle. The second equation directly conserves $p_\theta = mr^2\dot{\theta}$.

WHEN ACCELERATIONS ARE DETERMINED: The accelerations appear with the matrix

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}.$$

If W is invertible, the equations locally determine $\ddot{q}$ as a function of $(q, \dot{q}, t)$. For a smooth Lagrangian, this gives an ordinary differential system to which local initial-value theory applies. Without

this *regularity*, Euler–Lagrange remains an implicit equation that may carry constraints. Invertibility, positivity, and minimality are distinct properties.

The momentum p_i is attached to a coordinate, but its conservation does not depend on a fortunate choice of notation. If L does not depend on q^i , then $\dot{p}_i = 0$: the absence of a variable from the description signals a transformation that leaves the action unchanged.

3. SYMMETRIES AND NOETHER’S THEOREM. *WHY* do momentum and angular momentum, computed by different expressions, remain constant in such similar circumstances? In one case, shifting the origin does not change the model; in the other, rotating it does not. Noether turns this resemblance into a calculation: an invariance of the action lets us read off a conserved quantity. A symmetry changes the perspective on the same drawing; but the charge depends on the action and its boundaries, not merely on the drawing’s appearance.

THEOREM 3.1 (NOETHER AT FIXED TIME): Consider an infinitesimal transformation $\delta q^i = \xi^i(q, t)$, without varying t . Suppose that, for every path,

$$\delta L = \frac{dB(q, t)}{dt}, \quad \delta \dot{q}^i = \frac{d}{dt} \xi^i(q(t), t).$$

The case $B = 0$ is strict invariance of the Lagrangian. Along every solution of the Euler–Lagrange equations, the conserved quantity is

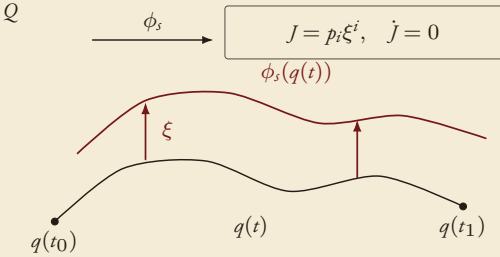
$$J = p_i \xi^i - B.$$

PROOF: The chain rule and $\dot{p}_i = \partial L / \partial q^i$ give

$$\delta L = \frac{\partial L}{\partial q^i} \xi^i + p_i \dot{\xi}^i = \frac{d}{dt} (p_i \xi^i).$$

Subtracting dB/dt concludes the proof. The symmetry condition is an identity before imposing the equations; conservation holds only once they are satisfied.

For a group of transformations ϕ_s of Q , the generator is $\xi = \partial_s \phi_s|_0$. The figure illustrates this strict case. Transformed curves may have different endpoints: a symmetry is not the fixed-endpoint variation used to establish Euler–Lagrange.



A strict symmetry at fixed time maps solutions to solutions. Its generator gives $J = p_i \xi^i$; the endpoints are transported as well.

For $L = m|\dot{q}|^2/2 - V(q)$, a translation in direction e_j , when V allows it, gives $\xi = e_j$ and $J = p_j$. A rotation about e_3 , for a potential invariant under that rotation, gives $\xi = e_3 \times q$ and $J = (q \times p)_3$. Conservation of p_θ in the preceding article is this same property in polar coordinates.

WHAT A BOUNDARY TERM CHANGES: For the free particle, take $q \mapsto q + \varepsilon ut$, with u constant. Here the generator depends on time, as the theorem allows: $\xi = ut$ and $\dot{\xi} = u$. We find

$$\delta L = m\dot{q} \cdot u = \frac{d}{dt}(mu \cdot q), \quad J = u \cdot (tp - mq).$$

This quantity is constant for every direction u . Together with constant p , it recovers affine motion. Omitting B would have given $u \cdot tp$, which is generally not conserved: an action invariant up to a boundary term does not yield exactly the charge of the strict case.

TIME AND ENERGY: A time translation is not simply a change of q at fixed time. Its effect can be checked directly. Define the Lagrangian energy

$$E_L = p_i \dot{q}^i - L.$$

Differentiating along a solution, the $\ddot{q}$ terms cancel and

$$\frac{dE_L}{dt} = \left(\dot{p}_i - \frac{\partial L}{\partial q^i} \right) \dot{q}^i - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}.$$

An autonomous Lagrangian therefore conserves E_L . For the natural model, $E_L = T + V$; this equality is not the general definition of energy. Noether's first theorem extended to time transformations recovers this balance, which we have just calculated without assuming that extension.

Noether's second theorem concerns symmetries with functional parameters and the resulting identities between equations; we will return to it with gauge theory. It does not remove the need to study boundary charges. Even here, conserving a quantity requires knowing exactly what the transformation leaves unchanged.

4. LAGRANGIANS WITH CONSTRAINTS. *THE* angle of a pendulum suffices to describe its motion, but does not directly give the force exerted by its rod. Two descriptions serve two needs: tracking only the allowed displacements, or retaining the reaction that enforces them. The first resembles an architectural plan already fitted to the terrain; the plan removes the need to redraw the constraint, without abolishing the forces required to respect it.

DEFINITION 4.1: A *holonomic* constraint involves positions and time. Write $f_\alpha(q, t) = 0$, for $\alpha = 1, \dots, k$. Assume $D_q f$ has rank k on their common zero set. At fixed time they define a submanifold Q_t of codimension k . A virtual displacement δq satisfies

$$D_q f \cdot \delta q = 0.$$

The reactions are called *ideal* if their virtual work $R \cdot \delta q$ vanishes for all such displacements.

Ideality is a physical assumption. It requires R to be a linear combination of the normals $\nabla_q f_\alpha$. In the Newtonian model, d'Alembert's principle thus reads $(m\ddot{q} - F) \cdot \delta q = 0$, summed over the particles. The applied forces F here exclude the reactions.

RETAINING THE NORMALS: With the sign chosen in the augmented action

$$S_\lambda = \int \left(L + \sum_\alpha \lambda_\alpha f_\alpha \right) dt,$$

varying the multipliers imposes $f_\alpha = 0$; varying q gives

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q^i}.$$

The right-hand side is the reaction. The λ_α are its coefficients in the chosen family of normals, not necessarily its Cartesian components. This sign corresponds to $L + \lambda f$; the optimization convention $f - \lambda g$ renames the multipliers without changing the principle.

THE PENDULUM IN BOTH DESCRIPTIONS: Take y upward, a length $\ell > 0$, and an angle θ measured from the downward vertical. The parametrization

$$x = \ell \sin \theta, \quad y = -\ell \cos \theta$$

builds the constraint into $L = m(\dot{x}^2 + \dot{y}^2)/2 - mgy$. It gives

$$L_{\text{red}} = \frac{m\ell^2}{2}\dot{\theta}^2 + mg\ell \cos \theta, \quad \ddot{\theta} + \frac{g}{\ell} \sin \theta = 0.$$

One equation suffices for the motion. To recover the reaction, instead retain $q = (x, y)$ and

$$f = \frac{1}{2}(x^2 + y^2 - \ell^2), \quad R = \lambda q.$$

The Cartesian equations are $m\ddot{x} = \lambda x$ and $m\ddot{y} = -mg + \lambda y$. Their tangential projection recovers the equation for θ . Differentiating $q \cdot q = \ell^2$ gives $q \cdot \ddot{q} = -|\dot{q}|^2 = -\ell^2 \dot{\theta}^2$; the radial projection then gives

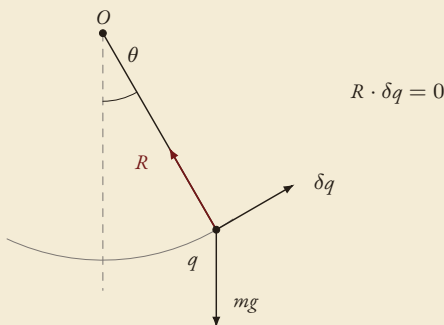
$$-m\ell^2 \dot{\theta}^2 = mg\ell \cos \theta + \lambda \ell^2.$$

Thus the inward component is

$$\mathcal{T} = -\lambda \ell = m(\ell \dot{\theta}^2 + g \cos \theta).$$

For a rod, this component is signed. For a string, $\mathcal{T} \geq 0$ is required to maintain the taut-string model: a string cannot push. The reaction could also be reconstructed from the reduced solution using Newton's equations; multipliers make the calculation explicit without being the only route.

At q , the allowed displacement runs along the circle; the ideal reaction is perpendicular to it. The drawing isolates this geometry, which eliminates R from the tangential projection.



Pendulum at $\theta = \pi/6$: the reaction shown points toward the support. Its length is illustrative; a rod also allows a reaction of the opposite sign. The vector δq is virtual, at fixed time.

WHAT REDUCTION ASSUMES: If the constraint depends on time, the configurations form Q_t rather than a single fixed manifold. An admissible motion satisfies $D_q f \cdot \dot{q} + \partial_t f = 0$, whereas a virtual displacement keeps t fixed. An ideal reaction can therefore supply actual power: moving the support can transfer energy.

Rolling without slipping may instead impose velocity relations that cannot be integrated into configuration equations. These *nonholonomic* constraints require the Lagrange–d’Alembert principle; inserting them naively into the action with multipliers can produce different dynamics. Restricting displacements is also distinct from taking a quotient by a symmetry. A constraint specifies what may vary; it does not yet decide whether stationary motion minimizes the action.

5. STATIONARITY, MINIMA, AND THE SECOND VARIATION. *EXTENDING* the observation time can turn an action minimum into a saddle point without changing either the force or the equation of motion. The harmonic oscillator suffices to show this. Just as a path on a surface may cease to be shortest when extended, a trajectory loses a variational property beyond a certain time. The image has a limit: this loss is not an instability of the oscillatory motion.

AN EXACTLY QUADRATIC DIFFERENCE: Fix $m, \omega > 0$ and the interval $[0, \Delta t]$. For

$$L = \frac{m}{2}(\dot{q}^2 - \omega^2 q^2),$$

Euler–Lagrange gives $\ddot{q} + \omega^2 q = 0$. Let q_* be a solution joining the prescribed endpoints. If η vanishes at both ends, the linear term in $S[q_* + \eta] - S[q_*]$ is

$$m \int (\dot{q}_* \dot{\eta} - \omega^2 q_* \eta) dt = -m \int (\ddot{q}_* + \omega^2 q_*) \eta dt = 0.$$

What remains is exactly

$$S[q_* + \eta] - S[q_*] = \frac{m}{2} \int_0^{\Delta t} (\dot{\eta}^2 - \omega^2 \eta^2) dt.$$

The *second variation* $\delta^2 S[q_*](\eta, \eta)$ is the second derivative of $S[q_* + \varepsilon \eta]$ at zero: here it equals twice the last right-hand side. No higher-order term obscures its sign.

CALCULATING THE THRESHOLD: The Poincaré inequality on $H_0^1([0, \Delta t])$, used here as a known result from analysis, states

$$\int \eta^2 dt \leq \left(\frac{\Delta t}{\pi} \right)^2 \int \dot{\eta}^2 dt.$$

It applies in particular to our smooth variations. If $\Delta t < \pi/\omega$, the action difference is therefore bounded below by

$$\frac{m}{2} \left(1 - \frac{\omega^2 \Delta t^2}{\pi^2} \right) \int \dot{\eta}^2 dt > 0 \quad (\eta \neq 0).$$

The trajectory is the unique minimizer for these endpoints; the estimate controls all directions at once.

To see what fails next, take $\eta(t) = a \sin(n\pi t/\Delta t)$, with $n \geq 1$. The integral of each squared trigonometric function is $\Delta t/2$, giving

$$S[q_* + \eta] - S[q_*] = \frac{ma^2}{4} \left(\frac{n^2 \pi^2}{\Delta t} - \omega^2 \Delta t \right).$$

If $\Delta t > \pi/\omega$, the $n = 1$ mode lowers the action, while a sufficiently high mode raises it. Arbitrarily small amplitudes give both signs: every solution to the stated problem is then a saddle point.

At the threshold $\Delta t = \pi/\omega$, the general solution $q(t) = q_0 \cos(\omega t) + b \sin(\omega t)$ requires $q_1 = -q_0$. If this condition holds, every b gives the same action, and Poincaré ensures non-strict minima. Otherwise, no solution joins the endpoints. The first zero mode expresses this meeting of neighboring trajectories; it gives the first conjugate time in this example.

WHAT A HESSIAN ALONE CANNOT TELL US: Legendre's necessary condition for a weak local minimum requires $L_{\dot{q}\dot{q}} \geq 0$ in the classical regular setting. Even strict positivity is insufficient: here this derivative always equals $m > 0$, including when the action has a negative direction. The full criterion involves the path and boundary conditions. For a general functional, a second variation positive in every direction does not by itself provide uniform control of the remainder; here the quadratic calculation and Poincaré supplied what was needed.

The geometric connection is through the energy $\frac{1}{2} \int g(\dot{q}, \dot{q}) dt$, whose critical points are affinely parametrized geodesics, studied in Riemannian geometry in D2. The length $\int \sqrt{g(\dot{q}, \dot{q})} dt$, by contrast, is invariant under reparametrization: its velocity Hessian is degenerate in the tangential direction. Passing from energy to length therefore changes the variational setting. Finding a stationary trajectory requires an equation; claiming that it is optimal also requires specifying what is being compared.

6. BRACHISTOCHROME AND CATENARY: ONE IDENTITY. A bead seeks the fastest path; a suspended chain settles into equilibrium. The first traces a cycloid, the second a hyperbolic cosine. Yet the two calculations share an identity. The same syntactic rule applies to two different statements: it reduces the equations without making travel time and potential energy interchangeable.

BELTRAMI'S IDENTITY: For an integrand $F(y, y')$ with no explicit dependence on x , every Euler–Lagrange solution satisfies

$$F - y'F_{y'} = C.$$

Indeed,

$$\frac{d}{dx}(F - y'F_{y'}) = y' \left(F_y - \frac{d}{dx}F_{y'} \right) = 0.$$

The absence of explicit x supplies a first integral. This is the energy calculation up to a sign, but the independent variable is now a horizontal coordinate: F need not be a mechanical Lagrangian.

DESCENDING AS FAST AS POSSIBLE: A bead starts from rest at A and slides without friction in uniform gravity. Taking y as depth below A , energy gives $v = \sqrt{2gy}$. For a path represented in its interior by a graph $y(x) > 0$, the travel time is

$$\mathcal{T}[y] = \int_0^{x_B} \sqrt{\frac{1 + y'^2}{2gy}} dx.$$

Beltrami gives a constant $F - y'F_{y'} = 1/\sqrt{2gy(1 + y'^2)}$, hence

$$y(1 + y'^2) = 2a, \quad a > 0.$$

The parametrization

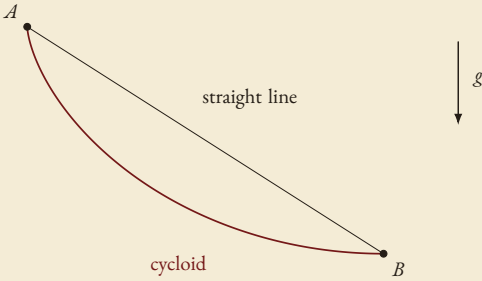
$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta), \quad 0 < \theta < 2\pi,$$

satisfies it: $y' = \sin \theta / (1 - \cos \theta) = \cot(\theta/2)$. The curve is a *cycloid*. For a prescribed endpoint to the right and below, the two coordinates fix a and θ_B .

The starting point has a vertical tangent and the integrand is singular at $y = 0$. The preceding equations therefore concern the interior; the travel time from the start is defined by an improper integral. Since $ds = 2a \sin(\theta/2)d\theta$ and $v = 2\sqrt{ga} \sin(\theta/2)$, their quotient simplifies:

$$\mathcal{T}_{\text{cycl}} = \theta_B \sqrt{a/g}.$$

The global optimality of this arc among admissible rectifiable curves joining the endpoints, with positive depth except at the start and finite travel time, is the classical brachistochrone result, stated here without proof. Solving Euler–Lagrange alone does not prove it.



For $B = (\pi a, 2a)$ in depth coordinates, a genuine cycloidal arc and a straight segment share their endpoints. The curve gains speed earlier.

The figure corresponds to $\theta_B = \pi$. The straight segment has length $a\sqrt{\pi^2 + 4}$ and tangential acceleration $2g/\sqrt{\pi^2 + 4}$. Starting from rest gives

$$\mathcal{T}_{\text{straight}} = \sqrt{\pi^2 + 4} \sqrt{a/g} > \pi \sqrt{a/g} = \mathcal{T}_{\text{cycl}}.$$

This comparison is calculated; it illustrates, without replacing, the optimality result among all curves.

SUSPENDING A CHAIN: Now take z upward. A uniform chain of linear mass density ρ minimizes its potential energy $\rho g \int z \sqrt{1 + z'^2} dx$ at fixed length. A constant multiplier μ gives the integrand

$$F = (\rho g z + \mu) \sqrt{1 + z'^2}.$$

Setting $b = \mu/(\rho g)$, Beltrami becomes

$$\frac{z + b}{\sqrt{1 + z'^2}} = a.$$

On the hanging branch $a > 0$, write $z + b = a \cosh u$. The equation gives $z'^2 = \sinh^2 u$, then $u' = \pm 1/a$ wherever $\sinh u \neq 0$. Smooth extension through the lowest point yields

$$z = a \cosh\left(\frac{x - x_0}{a}\right) - b.$$

This is the *catenary*. The endpoints and length determine its parameters. For attachments with distinct horizontal coordinates and a length exceeding their separation, the usual minimizing characterization is a classical result stated without proof; the identity has constructed the candidate here, not a general existence proof.

The two profiles differ because their functionals differ. What they share is the quantity conserved through an absence of explicit dependence. In mechanics, the combination $p_i \dot{q}^i - L$ will invite us to choose momenta as variables: the Legendre transformation will specify when this alternative description retains all the information.

FURTHER READING

- **Noether, E.**, *Invariante Variationsprobleme (1918)*, English translation by M. A. Tavel. The foundational text distinguishes groups with constant parameters from symmetries with functional parameters. Read the statements while separating conservation on solutions from identities between equations.
- **Arnold, V. I.**, *Mathematical Methods of Classical Mechanics*, 2nd edition, Springer, 1989. Mechanics becomes a geometric study of configurations and phases. The variational and Hamiltonian developments extend the pendulum calculations and the question of conjugate points treated here.
- **Goldstein, H., Poole, C. & Safko, J.**, *Classical Mechanics*, 3rd edition, Addison-Wesley, 2002. The first two chapters bring together variational principles, generalized coordinates, and constraints. Revisit the virtual-work assumptions before using multipliers.
- **Tong, D.**, *Classical Dynamics*, Chapter 2. Accessible notes tracing the Lagrangian formulation and symmetries. The examples let readers compare Newtonian and variational calculations without attributing initial-value uniqueness to a two-endpoint problem.
- **Liberzon, D.**, *Calculus of Variations and Optimal Control Theory*, Princeton University Press, 2012. Chapter 2 distinguishes necessary and sufficient conditions and types of minima; §§2.3.4 and 2.6.2 extend Beltrami's identity and the study of the second variation.

RESEARCH QUESTION

Recognizing variational dynamics. Which equations of motion can come from an action, and what freedom remains in this choice? Dissipation alone does not decide. For $\gamma > 0$, the Lagrangian

$$L = e^{\gamma t} \frac{m}{2} (\dot{q}^2 - \omega^2 q^2)$$

gives $d(me^{\gamma t}\dot{q})/dt + me^{\gamma t}\omega^2 q = 0$, the damped oscillator $\ddot{q} + \gamma\dot{q} + \omega^2 q = 0$. Yet its mechanical energy $E_{\text{mec}} = m(\dot{q}^2 + \omega^2 q^2)/2$ satisfies $\dot{E}_{\text{mec}} = -m\gamma\dot{q}^2$. The action depends explicitly on time; it does not turn this dissipated energy into a conserved quantity.

The inverse problem of the calculus of variations studies the conditions for such a reconstruction. This example is solved; it is not an open conjecture. To continue, fix the class of allowed Lagrangians, the regularity, and the variables retained, then ask when two actions produce the same trajectories. Adding a total derivative $dF(q, t)/dt$ already gives one freedom, since it does not change the variation at fixed endpoints. Understanding how far this freedom extends opens a structural question rather than a mere search for a formula.