



**Articulated Course**  
**in**  
**Mathematics**  
**III**  
**Graduate Mathematics**  
FROM FUNCTIONAL ANALYSIS TO GALOIS THEORY

**William de France**

CLEMENTINIUM EDITIONS

*Clem*



# GRADUATE MATHEMATICS

*There are no solved problems; there are only problems that are more or less solved.*

HENRI POINCARÉ

**Articulated Course**  
**in**  
**Mathematics**

III

**Graduate Mathematics**

FROM FUNCTIONAL ANALYSIS TO GALOIS THEORY

**William de France**

CLEMENTINIUM EDITIONS

Copyright © 2026 William de France

CLEMENTINIUM EDITIONS

Collection ARTICULATED COURSE IN MATHEMATICS

All rights reserved.

## FOREWORD

*This volume gathers the areas of graduate mathematics in four hundred and fifty-five articles, arranged in eighty-six chapters. It addresses the student who has crossed the sixteen parts of Foundations and now begins to deepen that knowledge: previously acquired structures become powerful enough to express profound patterns.*

*The thesis of the Articulated Course in Mathematics is that mathematics is the generative grammar by which reality composes itself. If Foundations taught that grammar (groups, vector spaces, distances, measure), this volume is the moment when the syntax grows rich enough to formulate complex sentences. Baire's theorem shows that a complete space possesses a solidity no countable accumulation can erode: completeness is a lever that turns pointwise arguments into uniform results. General topology retains from a metric space what persists once the distance is forgotten, as a sculptor retains from a block only the form. Galois theory reveals the hidden symmetries of algebraic equations. Distributions give meaning to derivatives of functions that possess none in the classical sense. At this level, the seven recurring themes of the collection (duality, symmetry, local-global, approximation, linearization, quotient, functoriality) are no longer intuitions: they are tools of proof.*

*The architecture remains that of the entire collection: the article is the fundamental unit. Each article isolates one idea and develops it in two hundred and fifty to eight hundred words. The same quotient construction that creates  $\mathbb{Z}/n\mathbb{Z}$  in arithmetic creates the quotient topology in geometry and the space  $L^p$  in analysis. The same approximation principle that governs Cauchy sequences governs simple functions in measure theory. To recognize these echoes from one domain to another is to pass from knowledge of the parts to a vision of the whole.*

*Reading is the first gesture. But understanding mathematics requires practicing it, and this volume accompanies every chapter with a program of exercises designed so that the reader is never left stranded. After each article, short practice exercises put the idea just presented into immediate practice.*

*At the end of each chapter, studies in methods teach the fundamental techniques – computing a spectrum, writing a weak formulation, the Galerkin method, the Lax-Milgram theorem, Céa’s lemma, solving a Riemann problem, entropy conditions, computing curvature, the Jacobi equation – each introduced by a lesson, then applied step by step. A recap gathers these methods onto a single page. Finally, guided expeditions lead the reader, in stages, toward results and research questions: the Einstein equations, the Riemann hypothesis, the Furstenberg-Szemerédi theorem, Connes’ noncommutative geometry, the Navier-Stokes Millennium Problem, the pinched sphere theorem, Calabi-Yau geometry. The reports distinguish solved steps, results stated without proof, and open questions. Solutions are printed upside down at the foot of each exercise: the reader can attempt the exercise without seeing the answer, then consult it by turning the book over.*

*Each chapter closes with a Further Reading box pointing to the original literature and standard references: Banach, Brezis, Rudin for functional analysis, Lang and Bourbaki for algebra, Hatcher and Lee for topology and manifolds, Hartshorne and Shafarevich for algebraic geometry. Half a page of brief annotations: these are not exhaustive bibliographies but signposts toward the natural continuation of each subject.*

*The ten companion objects of the collection continue their journey. The circle  $S^1$  is now a smooth manifold and a Lie group;  $\mathbb{Z}$  is a Euclidean domain whose ideals containing  $n\mathbb{Z}$  index the subgroups of  $\mathbb{Z}/n\mathbb{Z}$ ;  $\mathbb{Q}(\sqrt{2})$  is a number field whose Galois group is  $\mathbb{Z}/2\mathbb{Z}$ ; the torus  $\mathbb{T}^2$  is the quotient  $\mathbb{R}^2/\mathbb{Z}^2$ , the natural setting for Fourier analysis. These objects, glimpsed in the Elements, formalized at undergraduate level, serve here as laboratories where each new theory is tested.*

*The volume is organized in twenty-five parts. Functional analysis, measure, probability and statistics prepare differential equations, dynamical systems and Fourier analysis. Complex analysis, algebra and geometry lead to numerical methods, then to the chapters on working practices and exercises. The last three parts open toward advanced algebraic structures, geometry and topology, then advanced analysis, probability and arithmetic. It assumes mastery of the undergraduate content covered by the preceding volume. It prepares the reader for the two research volumes that follow: Correspondences, which leads from class field theory to the Langlands program, and Mathematical Physics, from symmetries to strings.*

*The mathematical conventions follow those of the French reference and the preceding volumes. English interval notation and decimal points are used. Conventions specific to a subject are stated where they first occur; the notation table also records the local conventions for Hermitian inner products.*

*Further reading has been adapted where verified English editions or resources serve the same purpose. Original sources and historical attributions are retained; section references concern the editions cited.*

*The student who opens this volume no longer seeks to learn mathematics: the student seeks to understand it. The difference is considerable. To understand is to see why a theorem is true, not merely that it is; to feel the necessity of a definition, not merely memorize it; to perceive, behind the diversity of domains, the unity of a grammar. This volume is designed to accompany that passage.*

## TO THE READER

*This course articulates in five volumes the grammar by which reality composes itself: over thirteen hundred articles of mathematics, from the first whole number to the Langlands correspondence and from the brachistochrone to the amplitubedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.*

*Each article develops a single idea, from its motivation to its interpretation. Definitions are framed. Results (propositions, theorems) are marked by a bold label; in volumes 2 and 3, major theorems are further framed in red. Proofs are present where they illuminate, sketched where full rigor would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The text distinguishes these three cases.*

*Ten mathematical objects run through the collection as recurring examples: the circle  $S^1$ , the integers  $\mathbb{Z}$ , the extension  $\mathbb{Q}(\sqrt{2})$ , the elliptic curve  $y^2 = x^3 - x$ , the group  $\mathrm{GL}_2$ , the function  $\zeta(s)$ , the space  $L^2$ , the ring  $K[X]$ , the symmetric group  $\mathfrak{S}_3$ , and the torus  $T^2$ . From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.*

*The figures, numbering more than eight hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colors. They do not replace proof; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.*

*In volumes I–III, each chapter extends into graded exercises (practice exercises, method studies, expeditions), whose detailed solutions are printed upside down at the foot of each exercise: the reader may attempt the exercise without seeing the answer, then read it by turning the book around. Whoever engages with them will find a teacher, never an examiner.*

*The reader may follow the linear path or take the cross-volume passages. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands program; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.*

*Further reading points to English editions where these can be identified, or to verified English alternatives with the same purpose. Historical attributions are retained; works available only in another language are identified as such. These readings extend the course rather than form part of its prerequisites.*

*Every error reported is an error corrected.*

**wdf@clementinium.com**

## NOTE ON THE LEVELS

*The five volumes follow a progression from the first mathematical objects to questions of research:*

- Elements (I): *secondary-school mathematics, from arithmetic to the first proofs.*
- Foundations (II): *the foundations of university mathematics.*
- Graduate Mathematics (III): *deeper study and preparation for research.*
- Correspondences (IV): *the research level, from class field theory to the Langlands program.*
- Mathematical Physics (V): *the research level, from symmetries to strings.*

*These levels guide the reader without imposing an academic timetable. Parts and chapters mark the stages of the journey; the pace depends on the reader's background and the questions they wish to explore.*

*Volumes I, II, and III successively build the prerequisites for the next volume. Readers may enter at the level for which they have the necessary background. Volumes IV and V share the foundation of Graduate Mathematics and may be read independently of each other. The bridges at the beginning of each volume explain the change in perspective.*

# NOTATIONS

|                                              |                                                                         |
|----------------------------------------------|-------------------------------------------------------------------------|
| $\mathbb{N}$                                 | <i>natural numbers</i>                                                  |
| $\mathbb{Z}$                                 | <i>integers</i>                                                         |
| $\mathbb{Q}$                                 | <i>rational numbers</i>                                                 |
| $\mathbb{R}$                                 | <i>real numbers</i>                                                     |
| $\mathbb{C}$                                 | <i>complex numbers</i>                                                  |
| $\in$                                        | <i>belongs to</i>                                                       |
| $\subseteq$                                  | <i>contained in</i>                                                     |
| $\cup, \cap, \emptyset$                      | <i>union, intersection, empty set</i>                                   |
| $\mathcal{P}(E)$                             | <i>power set of <math>E</math></i>                                      |
| $A \setminus B$                              | <i>set difference</i>                                                   |
| $\sqcup$                                     | <i>disjoint union</i>                                                   |
| $\forall, \exists$                           | <i>for all, there exists</i>                                            |
| $\Rightarrow, \Leftrightarrow$               | <i>implies, is equivalent to</i>                                        |
| $ x $                                        | <i>absolute value of <math>x</math></i>                                 |
| $\sqrt{x}$                                   | <i>square root of <math>x</math></i>                                    |
| $\leq, \geq$                                 | <i>less than or equal to, greater than or equal to</i>                  |
| $\pi$                                        | <i>ratio of circumference to diameter</i>                               |
| $\infty$                                     | <i>infinity</i>                                                         |
| $\sin, \cos, \tan$                           | <i>trigonometric functions</i>                                          |
| $\ln, \exp$                                  | <i>natural logarithm, exponential</i>                                   |
| $f'(x)$                                      | <i>derivative of <math>f</math> at <math>x</math></i>                   |
| $\int_a^b f(x) \, dx$                        | <i>integral of <math>f</math> from <math>a</math> to <math>b</math></i> |
| $\sum, \prod$                                | <i>sum, product</i>                                                     |
| $n!, \binom{n}{k}$                           | <i>factorial, binomial coefficient</i>                                  |
| $i$                                          | <i>imaginary unit, <math>i^2 = -1</math></i>                            |
| $\bar{z},  z , \arg(z)$                      | <i>conjugate, modulus, argument</i>                                     |
| $\operatorname{Re}(z), \operatorname{Im}(z)$ | <i>real part, imaginary part</i>                                        |
| $\rightarrow$                                | <i>tends to</i>                                                         |
| $x \mapsto f(x)$                             | <i>the map sending <math>x</math> to <math>f(x)</math></i>              |
| $\vec{u}$                                    | <i>vector <math>u</math></i>                                            |
| $f \circ g$                                  | <i>composition of <math>f</math> and <math>g</math></i>                 |
| $\ x\ $                                      | <i>norm of <math>x</math></i>                                           |

|                                                              |                                                                                        |
|--------------------------------------------------------------|----------------------------------------------------------------------------------------|
| $\langle x, y \rangle$                                       | <i>inner product of <math>x</math> and <math>y</math></i>                              |
| $\dim V$                                                     | <i>dimension of a vector space <math>V</math></i>                                      |
| $\ker f, \operatorname{im} f$                                | <i>kernel, image of a linear map</i>                                                   |
| $\det A$                                                     | <i>determinant of a matrix <math>A</math></i>                                          |
| $A^T$                                                        | <i>transpose of <math>A</math></i>                                                     |
| $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}_n$                 | <i>general linear, special linear, orthogonal groups</i>                               |
| $a \equiv b \pmod{n}$                                        | <i>congruence modulo <math>n</math></i>                                                |
| $\cong, \simeq$                                              | <i>isomorphic to, equivalent to</i>                                                    |
| $\oplus, \otimes$                                            | <i>direct sum, tensor product</i>                                                      |
| $\lim$                                                       | <i>limit</i>                                                                           |
| $\sup, \inf$                                                 | <i>supremum, infimum</i>                                                               |
| $\limsup, \liminf$                                           | <i>limit superior, limit inferior</i>                                                  |
| $O(\cdot), o(\cdot), \sim$                                   | <i>Landau notation, asymptotic equivalence</i>                                         |
| $\frac{d}{dx}, \frac{\partial}{\partial x}$                  | <i>ordinary derivative, partial derivative</i>                                         |
| $\nabla f$                                                   | <i>gradient of <math>f</math></i>                                                      |
| $\overline{A}$                                               | <i>closure of <math>A</math></i>                                                       |
| $L^p$                                                        | <i>Lebesgue space of exponent <math>p</math></i>                                       |
| $C^k, C^\infty$                                              | <i><math>k</math> times continuously differentiable, smooth</i>                        |
| $\mu, \lambda$                                               | <i>measure, Lebesgue measure</i>                                                       |
| $P(A)$                                                       | <i>probability of the event <math>A</math></i>                                         |
| $\mathbb{E}[X], \operatorname{Var}(X)$                       | <i>expectation, variance of a random variable</i>                                      |
| $\mathbf{1}_A$                                               | <i>indicator function of <math>A</math></i>                                            |
| $\mathcal{N}(m, \sigma^2)$                                   | <i>normal distribution with mean <math>m</math> and variance <math>\sigma^2</math></i> |
| $\mathcal{L}(E, F)$                                          | <i>bounded linear maps from <math>E</math> to <math>F</math></i>                       |
| $E^*, E^{**}$                                                | <i>topological dual, bidual of <math>E</math></i>                                      |
| $\sigma(T), \rho(T)$                                         | <i>spectrum, resolvent set of an operator <math>T</math></i>                           |
| $W^{k,p}(\Omega), H^s(\Omega)$                               | <i>Sobolev spaces</i>                                                                  |
| $\mathcal{D}'(\Omega)$                                       | <i>distributions on <math>\Omega</math></i>                                            |
| $\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)$      | <i>Schwartz space, tempered distributions</i>                                          |
| $\operatorname{Spec} A$                                      | <i>spectrum of a ring <math>A</math></i>                                               |
| $\operatorname{Hom}, \operatorname{Ext}, \operatorname{Tor}$ | <i>homological algebra functors</i>                                                    |
| $\wedge$                                                     | <i>exterior product</i>                                                                |
| $H^n$                                                        | <i>cohomology in degree <math>n</math></i>                                             |
| $\pi_1$                                                      | <i>fundamental group</i>                                                               |
| $\operatorname{Gal}(L/K)$                                    | <i>Galois group of the extension <math>L/K</math></i>                                  |
| $[L : K]$                                                    | <i>degree of the extension <math>L/K</math></i>                                        |

*Hermitian inner product conventions are stated locally: linear in the first variable in Chapter 4, antilinear in the first variable in Chapters 45 and 47. To compare formulas, exchange the arguments:  $\Phi(x, y) = \langle y, x \rangle$ .*

# Contents

|                                                 |              |
|-------------------------------------------------|--------------|
| <b>Foreword</b>                                 | <b>v</b>     |
| <b>To the Reader</b>                            | <b>viii</b>  |
| <b>Note on the Levels</b>                       | <b>x</b>     |
| <b>Notations</b>                                | <b>xi</b>    |
| <b>From Foundations to Graduate Mathematics</b> | <b>i</b>     |
| <br><b>I Functional Analysis</b>                | <br><b>2</b> |
| <b>1 Metric Spaces</b>                          | <b>3</b>     |
| Methods . . . . .                               | 22           |
| Expeditions . . . . .                           | 28           |
| <b>2 General Topology</b>                       | <b>37</b>    |
| Methods . . . . .                               | 52           |
| Expeditions . . . . .                           | 62           |
| <b>3 Normed Spaces</b>                          | <b>68</b>    |
| Methods . . . . .                               | 87           |
| Expeditions . . . . .                           | 95           |

|           |                                                               |            |
|-----------|---------------------------------------------------------------|------------|
| <b>4</b>  | <b>Hilbert Spaces</b>                                         | <b>102</b> |
|           | Methods . . . . .                                             | 122        |
|           | Expeditions . . . . .                                         | 132        |
| <b>5</b>  | <b>Topology of Function Spaces</b>                            | <b>139</b> |
|           | Methods . . . . .                                             | 155        |
|           | Expeditions . . . . .                                         | 162        |
| <b>6</b>  | <b>Convergence and Approximation</b>                          | <b>167</b> |
|           | Methods . . . . .                                             | 185        |
|           | Expeditions . . . . .                                         | 192        |
| <b>7</b>  | <b>Fundamental Theorems of Functional Analysis</b>            | <b>199</b> |
|           | Methods . . . . .                                             | 215        |
|           | Expeditions . . . . .                                         | 222        |
| <b>8</b>  | <b>Weak Topologies and Reflexivity</b>                        | <b>228</b> |
|           | Methods . . . . .                                             | 247        |
|           | Expeditions . . . . .                                         | 254        |
| <b>9</b>  | <b>Spectral Theory</b>                                        | <b>259</b> |
|           | Methods . . . . .                                             | 295        |
|           | Expeditions . . . . .                                         | 305        |
| <b>II</b> | <b>Differential Calculus</b>                                  | <b>312</b> |
| <b>10</b> | <b>Differential Calculus in Finite and Infinite Dimension</b> | <b>313</b> |
|           | Methods . . . . .                                             | 343        |
|           | Expeditions . . . . .                                         | 352        |

|                       |                                                |            |
|-----------------------|------------------------------------------------|------------|
| <b>III</b>            | <b>Convexity and Optimization</b>              | <b>360</b> |
| <b>11</b>             | <b>Convexity</b>                               | <b>361</b> |
| Methods . . . . .     |                                                | 380        |
| Expeditions . . . . . |                                                | 387        |
| <b>12</b>             | <b>Optimization</b>                            | <b>392</b> |
| Methods . . . . .     |                                                | 425        |
| Expeditions . . . . . |                                                | 431        |
| <b>IV</b>             | <b>Fixed Points and Applications</b>           | <b>435</b> |
| <b>13</b>             | <b>Fixed Points and Applications</b>           | <b>436</b> |
| Methods . . . . .     |                                                | 456        |
| Expeditions . . . . . |                                                | 463        |
| <b>V</b>              | <b>Measure and Integration</b>                 | <b>469</b> |
| <b>14</b>             | <b>Sigma-Algebras, Measures, Measurability</b> | <b>470</b> |
| Methods . . . . .     |                                                | 483        |
| Expeditions . . . . . |                                                | 488        |
| <b>15</b>             | <b>Constructing the Lebesgue Integral</b>      | <b>493</b> |
| Methods . . . . .     |                                                | 506        |
| Expeditions . . . . . |                                                | 513        |
| <b>16</b>             | <b>Convergence Theorems</b>                    | <b>519</b> |
| Methods . . . . .     |                                                | 535        |
| Expeditions . . . . . |                                                | 541        |
| <b>17</b>             | <b><math>L^p</math> Spaces</b>                 | <b>547</b> |
| Methods . . . . .     |                                                | 564        |
| Expeditions . . . . . |                                                | 571        |

|            |                                                   |            |
|------------|---------------------------------------------------|------------|
| <b>18</b>  | <b>Construction and Decomposition of Measures</b> | <b>576</b> |
|            | Methods . . . . .                                 | 593        |
|            | Expeditions . . . . .                             | 598        |
| <b>19</b>  | <b>Advanced Integration</b>                       | <b>604</b> |
|            | Methods . . . . .                                 | 621        |
|            | Expeditions . . . . .                             | 628        |
| <b>VI</b>  | <b>Probability and Processes</b>                  | <b>632</b> |
| <b>20</b>  | <b>Probability</b>                                | <b>633</b> |
|            | Methods . . . . .                                 | 671        |
|            | Expeditions . . . . .                             | 679        |
| <b>21</b>  | <b>Discrete Processes</b>                         | <b>684</b> |
|            | Methods . . . . .                                 | 701        |
|            | Expeditions . . . . .                             | 707        |
| <b>22</b>  | <b>Continuous Processes</b>                       | <b>711</b> |
|            | Methods . . . . .                                 | 725        |
|            | Expeditions . . . . .                             | 729        |
| <b>VII</b> | <b>Mathematical Statistics</b>                    | <b>733</b> |
| <b>23</b>  | <b>Statistical Models and Estimation</b>          | <b>734</b> |
|            | Methods . . . . .                                 | 769        |
|            | Expeditions . . . . .                             | 775        |
| <b>24</b>  | <b>Tests and Regression</b>                       | <b>779</b> |
|            | Methods . . . . .                                 | 812        |
|            | Expeditions . . . . .                             | 818        |

|             |                                          |            |
|-------------|------------------------------------------|------------|
| <b>VIII</b> | <b>Differential Equations</b>            | <b>822</b> |
| 25          | Ordinary Differential Equations          | 823        |
|             | Methods . . . . .                        | 849        |
|             | Expeditions . . . . .                    | 854        |
| 26          | Classical PDEs                           | 859        |
|             | Methods . . . . .                        | 875        |
|             | Expeditions . . . . .                    | 880        |
| 27          | Methods for PDEs                         | 886        |
|             | Methods . . . . .                        | 901        |
|             | Expeditions . . . . .                    | 906        |
| 28          | Further Topics in Differential Equations | 912        |
|             | Methods . . . . .                        | 926        |
|             | Expeditions . . . . .                    | 932        |
| <b>IX</b>   | <b>Dynamical Systems</b>                 | <b>936</b> |
| 29          | Continuous Dynamical Systems             | 937        |
|             | Methods . . . . .                        | 951        |
|             | Expeditions . . . . .                    | 958        |
| 30          | Discrete Dynamical Systems               | 964        |
|             | Methods . . . . .                        | 978        |
|             | Expeditions . . . . .                    | 984        |
| 31          | Stability and Bifurcations               | 989        |
|             | Methods . . . . .                        | 1003       |
|             | Expeditions . . . . .                    | 1009       |

|           |                                                     |             |
|-----------|-----------------------------------------------------|-------------|
| <b>X</b>  | <b>Distributions and Fourier</b>                    | <b>1015</b> |
| 32        | Fourier Series                                      | 1016        |
|           | Methods . . . . .                                   | 1034        |
|           | Expeditions . . . . .                               | 1042        |
| 33        | Distributions                                       | 1048        |
|           | Methods . . . . .                                   | 1064        |
|           | Expeditions . . . . .                               | 1070        |
| 34        | The Fourier Transform and Tempered<br>Distributions | 1076        |
|           | Methods . . . . .                                   | 1097        |
|           | Expeditions . . . . .                               | 1104        |
| 35        | Sobolev Spaces                                      | 1110        |
|           | Methods . . . . .                                   | 1127        |
|           | Expeditions . . . . .                               | 1131        |
| 36        | Applications of Fourier Analysis                    | 1137        |
|           | Methods . . . . .                                   | 1152        |
|           | Expeditions . . . . .                               | 1158        |
| <b>XI</b> | <b>Advanced Complex Analysis</b>                    | <b>1163</b> |
| 37        | Holomorphic Functions                               | 1164        |
|           | Methods . . . . .                                   | 1183        |
|           | Expeditions . . . . .                               | 1189        |
| 38        | Complex Integration                                 | 1196        |
|           | Methods . . . . .                                   | 1217        |
|           | Expeditions . . . . .                               | 1223        |
| 39        | Global Theory of Holomorphic Functions              | 1230        |

|                       |      |
|-----------------------|------|
| Methods . . . . .     | 1246 |
| Expeditions . . . . . | 1251 |

#### **40 Conformal Maps** **1257**

|                       |      |
|-----------------------|------|
| Methods . . . . .     | 1273 |
| Expeditions . . . . . | 1278 |

#### **41 Special Functions and Riemann Surfaces** **1282**

|                       |      |
|-----------------------|------|
| Methods . . . . .     | 1298 |
| Expeditions . . . . . | 1303 |

### **XII Advanced Linear Algebra** **1310**

#### **42 Duality and Multilinearity** **1311**

|                       |      |
|-----------------------|------|
| Methods . . . . .     | 1324 |
| Expeditions . . . . . | 1329 |

#### **43 Modules over a Ring** **1333**

|                       |      |
|-----------------------|------|
| Methods . . . . .     | 1347 |
| Expeditions . . . . . | 1352 |

#### **44 Structural Decompositions** **1355**

|                       |      |
|-----------------------|------|
| Methods . . . . .     | 1374 |
| Expeditions . . . . . | 1379 |

#### **45 Forms and Bilinear Algebra** **1382**

|                       |      |
|-----------------------|------|
| Methods . . . . .     | 1398 |
| Expeditions . . . . . | 1404 |

### **XIII Bilinear Geometry** **1407**

#### **46 Bilinear and Quadratic Forms** **1408**

|                   |      |
|-------------------|------|
| Methods . . . . . | 1423 |
|-------------------|------|

|                                                 |             |
|-------------------------------------------------|-------------|
| Expeditions . . . . .                           | 1428        |
| <b>47 Euclidean and Hermitian Spaces</b>        | <b>1432</b> |
| Methods . . . . .                               | 1452        |
| Expeditions . . . . .                           | 1456        |
| <b>48 Geometry of Forms</b>                     | <b>1459</b> |
| Methods . . . . .                               | 1479        |
| Expeditions . . . . .                           | 1484        |
| <b>XIV Abstract Algebra</b>                     | <b>1488</b> |
| <b>49 Group Theory</b>                          | <b>1489</b> |
| Methods . . . . .                               | 1511        |
| Expeditions . . . . .                           | 1514        |
| <b>50 Rings and Ideals</b>                      | <b>1518</b> |
| Methods . . . . .                               | 1537        |
| Expeditions . . . . .                           | 1542        |
| <b>51 Fields, Extensions, and Separability</b>  | <b>1546</b> |
| Methods . . . . .                               | 1570        |
| Expeditions . . . . .                           | 1575        |
| <b>XV Galois Theory</b>                         | <b>1578</b> |
| <b>52 Galois Theory</b>                         | <b>1579</b> |
| Methods . . . . .                               | 1596        |
| Expeditions . . . . .                           | 1606        |
| <b>XVI Structures and Commutative Alg.</b>      | <b>1614</b> |
| <b>53 Products, Representations, Categories</b> | <b>1615</b> |

|                                                              |             |
|--------------------------------------------------------------|-------------|
| Methods . . . . .                                            | 1630        |
| Expeditions . . . . .                                        | 1638        |
| <b>54 Commutative Algebra</b>                                | <b>1643</b> |
| Methods . . . . .                                            | 1662        |
| Expeditions . . . . .                                        | 1672        |
| <b>55 Towards Algebraic Geometry</b>                         | <b>1677</b> |
| Methods . . . . .                                            | 1688        |
| Expeditions . . . . .                                        | 1699        |
| <b>XVII Topology and Diff. Geometry</b>                      | <b>1705</b> |
| <b>56 Covering Spaces and the Fundamental Group</b>          | <b>1706</b> |
| Methods . . . . .                                            | 1722        |
| Expeditions . . . . .                                        | 1731        |
| <b>57 Differentiable Manifolds</b>                           | <b>1738</b> |
| Methods . . . . .                                            | 1756        |
| Expeditions . . . . .                                        | 1765        |
| <b>58 Vector Fields, Differential Forms, and Integration</b> | <b>1770</b> |
| Methods . . . . .                                            | 1794        |
| Expeditions . . . . .                                        | 1802        |
| <b>XVIII Algebraic Geometry</b>                              | <b>1809</b> |
| <b>59 Affine Varieties</b>                                   | <b>1810</b> |
| Methods . . . . .                                            | 1825        |
| Expeditions . . . . .                                        | 1835        |

|                       |                                            |             |
|-----------------------|--------------------------------------------|-------------|
| <b>60</b>             | <b>Projective Varieties</b>                | <b>1842</b> |
| Methods . . . . .     |                                            | 1859        |
| Expeditions . . . . . |                                            | 1868        |
| <b>61</b>             | <b>Towards Schemes</b>                     | <b>1875</b> |
| Methods . . . . .     |                                            | 1888        |
| Expeditions . . . . . |                                            | 1897        |
| <b>XIX</b>            | <b>Numerical Analysis</b>                  | <b>1903</b> |
| <b>62</b>             | <b>Approximation and Interpolation</b>     | <b>1904</b> |
| Methods . . . . .     |                                            | 1922        |
| Expeditions . . . . . |                                            | 1931        |
| <b>63</b>             | <b>Linear Systems and Eigenvalues</b>      | <b>1938</b> |
| Methods . . . . .     |                                            | 1955        |
| Expeditions . . . . . |                                            | 1967        |
| <b>64</b>             | <b>Integration and Nonlinear Equations</b> | <b>1974</b> |
| Methods . . . . .     |                                            | 1987        |
| Expeditions . . . . . |                                            | 1999        |
| <b>65</b>             | <b>Numerical Differential Equations</b>    | <b>2008</b> |
| Methods . . . . .     |                                            | 2023        |
| Expeditions . . . . . |                                            | 2033        |
| <b>XX</b>             | <b>Scientific Computing</b>                | <b>2040</b> |
| <b>66</b>             | <b>Mathematical Modeling</b>               | <b>2041</b> |
| Methods . . . . .     |                                            | 2053        |
| Expeditions . . . . . |                                            | 2063        |
| <b>67</b>             | <b>Numerical Simulation</b>                | <b>2069</b> |

|                                                |                 |
|------------------------------------------------|-----------------|
| Methods . . . . .                              | 2087            |
| Expeditions . . . . .                          | 2100            |
| <b>68 Case Studies</b>                         | <b>2108</b>     |
| Methods . . . . .                              | 2122            |
| Expeditions . . . . .                          | 2133            |
| <br><b>XXI Reading and Working</b>             | <br><b>2142</b> |
| <b>69 How to Read Graduate Mathematics</b>     | <b>2143</b>     |
| <b>70 How to Study Graduate Mathematics</b>    | <b>2150</b>     |
| <br><b>XXII Transition Exercises</b>           | <br><b>2157</b> |
| <b>71 Exercises in Analysis</b>                | <b>2158</b>     |
| <b>72 Exercises in Algebra</b>                 | <b>2166</b>     |
| <b>73 Mixed Exercises</b>                      | <b>2173</b>     |
| <br><b>XXIII Advanced Algebraic Structures</b> | <br><b>2181</b> |
| <b>74 Category Theory</b>                      | <b>2182</b>     |
| Methods . . . . .                              | 2203            |
| Expeditions . . . . .                          | 2207            |
| <b>75 Homological Algebra</b>                  | <b>2213</b>     |
| Methods . . . . .                              | 2239            |
| Expeditions . . . . .                          | 2243            |
| <b>76 Representations of Finite Groups</b>     | <b>2248</b>     |
| Methods . . . . .                              | 2267            |

|                       |      |
|-----------------------|------|
| Expeditions . . . . . | 2271 |
| 77 Lie Algebras       | 2275 |
| Methods . . . . .     | 2290 |
| Expeditions . . . . . | 2294 |
| 78 Lie Groups         | 2298 |
| Methods . . . . .     | 2314 |
| Expeditions . . . . . | 2319 |

**XXIV Differential Geometry and Topology** **2322**

|                              |      |
|------------------------------|------|
| 79 Singular Homology         | 2323 |
| Methods . . . . .            | 2340 |
| Expeditions . . . . .        | 2345 |
| 80 Space Curves and Surfaces | 2348 |
| Methods . . . . .            | 2363 |
| Expeditions . . . . .        | 2368 |
| 81 Riemannian Geometry       | 2373 |
| Methods . . . . .            | 2391 |
| Expeditions . . . . .        | 2398 |

**XXV Advanced Analysis, Probability, and Arithmetic** **2403**

|                       |      |
|-----------------------|------|
| 82 Itô Calculus       | 2404 |
| Methods . . . . .     | 2423 |
| Expeditions . . . . . | 2429 |

|                       |                                                     |             |
|-----------------------|-----------------------------------------------------|-------------|
| <b>83</b>             | <b>Banach Algebras and Advanced Spectral Theory</b> | <b>2433</b> |
| Methods . . . . .     |                                                     | 2448        |
| Expeditions . . . . . |                                                     | 2453        |
| <b>84</b>             | <b>Ergodic Theory</b>                               | <b>2456</b> |
| Methods . . . . .     |                                                     | 2471        |
| Expeditions . . . . . |                                                     | 2476        |
| <b>85</b>             | <b>Nonlinear PDEs</b>                               | <b>2479</b> |
| Methods . . . . .     |                                                     | 2493        |
| Expeditions . . . . . |                                                     | 2498        |
| <b>86</b>             | <b>Advanced Elementary Number Theory</b>            | <b>2504</b> |
| Methods . . . . .     |                                                     | 2519        |
| Expeditions . . . . . |                                                     | 2523        |
|                       | <b>At the End of This Volume</b>                    | <b>2526</b> |
|                       | <b>Bibliography</b>                                 | <b>2527</b> |
|                       | <b>Index</b>                                        | <b>2531</b> |



## FROM FOUNDATIONS TO GRADUATE MATHEMATICS

*At the undergraduate level, the path was marked: definition, theorem, proof, exercise. At the graduate level, one must often find the path oneself. This is the most important change, and it concerns the way of seeing rather than the theorems.*

*The first tools are in place: vector spaces, groups, polynomials, differential and integral calculus, series, and the beginnings of topology. Graduate mathematics extends them into territories where finite dimension no longer suffices, where topology becomes a choice, where symmetries hide in increasingly refined algebraic structures. The Lebesgue integral, Fourier series, and holomorphic functions will be constructed in this volume: they will broaden the ways we pass to limits and represent functions.*

*Functional analysis will put operators to work on infinite-dimensional spaces. Distributions will give meaning to derivatives of functions that have none in the classical sense. Galois theory will reveal the invisible symmetries of polynomial equations. And manifolds will replace open subsets of  $\mathbb{R}^n$  with curved spaces that can only be observed locally.*

*Reading an article, following a seminar, identifying the right question: these skills, harder to teach than theorems, are what make a mathematician. This volume is devoted to their acquisition.*

# Part I

## Functional Analysis

*Two chapters of topology open the volume: metric spaces, where completeness and compactness are defined, then topological spaces, defined by their open sets alone. Functional analysis then studies infinite-dimensional vector spaces equipped with a norm, together with the continuous linear maps between them. Banach and Hilbert spaces come next, followed by the four fundamental theorems: Hahn-Banach, Banach-Steinhaus, the open mapping theorem, and the closed graph theorem. Weak topologies and the spectral theory of bounded operators complete the framework. Completeness is the constant lever: it turns pointwise estimates into uniform results.*

## CHAPTER I

### METRIC SPACES

*Sequences, limits and continuity were defined on  $\mathbb{R}$  with the absolute value; on  $\mathbb{R}^n$ , on spaces of functions, on sequences, the same construction only requires a distance, and that is the only thing a metric space retains. This chapter installs this setting and the four notions that follow from it. Open sets, closed sets and neighborhoods translate the distance into topological language. Completeness, where every Cauchy sequence converges, is the property that allows limits to be built without being known, whence Banach's fixed point and Baire's theorem. Compactness, where every sequence has a cluster point, is the one that guarantees extrema. Connectedness, finally, says what a space in one piece is, and founds the intermediate value theorem.*

*Articles in this chapter:*

1. *Distance and Metric Spaces — A metric space is a set equipped with a distance satisfying separation, symmetry, and the triangle inequality.*
2. *Open Sets, Closed Sets, Neighborhoods — Open sets, closed sets, and neighborhoods are defined by balls.*
3. *Completeness — A complete metric space contains the limits promised by its Cauchy sequences.*
4. *Compactness — Compactness turns infinite data into finite or convergent data.*
5. *Connectedness — Connectedness formalizes the idea of being in one piece.*

#### **I. DISTANCE AND METRIC SPACES.**

Two continuous functions on  $[0, 1]$  can be close in one sense and far apart in another: close if the area between their graphs is small, far apart if their values differ sharply at some points, as a very

narrow, very tall continuous spike shows (small area yet large maximal gap). Behind this ambiguity lies a fundamental question: what does *being close* mean in a set that is not  $\mathbb{R}^n$ ? The topology of  $\mathbb{R}^n$  learnt in the previous volume, built on the Euclidean distance, had nothing specific to  $\mathbb{R}^n$ . Once one isolates the properties of a distance that underpin the theory (separation, symmetry, triangle inequality), the same edifice can be erected on any set. The resulting framework, the metric space, is like an architect's blueprint: it draws the topology without fixing the materials.

**DEFINITION 1.1:** Let  $X$  be a set. A *distance* (or *metric*) on  $X$  is a map  $d: X \times X \rightarrow \mathbb{R}_+$  satisfying, for all  $x, y, z \in X$ :

1. *Separation*:  $d(x, y) = 0 \Leftrightarrow x = y$ .
2. *Symmetry*:  $d(x, y) = d(y, x)$ .
3. *Triangle inequality*:  $d(x, z) \leq d(x, y) + d(y, z)$ .

**DEFINITION 1.2:** A *metric space* is a pair  $(X, d)$  where  $X$  is a nonempty set and  $d$  a distance on  $X$ .

These three axioms suffice to generate an entire topology. Before seeing how, let us examine the diversity of situations they encompass.

**EXAMPLE:** (1)  $(\mathbb{R}^n, d_2)$ , where  $d_2(x, y) = \|x - y\|_2$ , is Euclidean space. (2)  $(C([a, b]), d_\infty)$ , where  $d_\infty(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|$ , measures the uniform gap between continuous functions. (3) The circle  $S^1$  and the torus  $T^2 = S^1 \times S^1$ , equipped with the distance induced by  $\mathbb{R}^n$ , are compact metric spaces that will reappear as domains for Fourier analysis. (4) The discrete distance, defined by  $d(x, y) = 1$  if  $x \neq y$  and  $d(x, x) = 0$ , makes any set into a metric space. (5) On  $\mathbb{R}^n$ , the distances  $d_1(x, y) = \sum |x_i - y_i|$  and  $d_\infty(x, y) = \max |x_i - y_i|$  are also distances.

**DEFINITION 1.3:** Let  $(X, d)$  be a metric space,  $x \in X$  and  $r > 0$ . The *open ball* of center  $x$  and radius  $r$  is  $B(x, r) = \{y \in X : d(x, y) < r\}$ . The *closed ball* is  $\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$ .

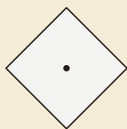
Open balls play the role of open intervals in  $\mathbb{R}$ : they serve as the elementary bricks from which the topology is built. A subset  $U$  is declared open if each of its points is the center of an open ball contained in  $U$ .

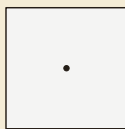
**PROPOSITION 1.4:** The open balls generate a topology on  $(X, d)$ : a set  $U \subset X$  is open if and only if for every  $x \in U$ , there exists  $r > 0$  such that  $B(x, r) \subset U$ .

*This result is stated without proof.*

Two distances on the same set may generate the same topology (the same open sets, the same convergences) while measuring gaps differently.

**DEFINITION 1.5:** Two distances  $d$  and  $d'$  on  $X$  are *topologically equivalent* if they define the same topology (the same open sets). A sufficient condition, called *bilipschitz equivalence* and strictly stronger, is the existence of constants  $\alpha, \beta > 0$  such that  $\alpha d \leq d' \leq \beta d$ .


 $d_1$ 

 $d_2$ 

 $d_\infty$ 

Open balls in  $\mathbb{R}^2$  for  $d_1, d_2, d_\infty$

**REMARK 1.6:** On  $\mathbb{R}^n$ , all distances arising from the norms  $\|\cdot\|_p$  ( $1 \leq p \leq \infty$ ) are equivalent: the balls change shape (square, disk, rotated square) but the open sets remain the same. This phenomenon is specific to finite dimension. In  $C([a, b])$ , the distance  $d_\infty$  and the distance  $d_1(f, g) = \int_a^b |f(t) - g(t)| dt$  (with  $a < b$ ) are not equivalent: uniform convergence is strictly stronger than convergence in the mean. The geometry of the space then depends genuinely on the choice of distance.

The notion of metric space thus isolates the minimal structure needed to speak of proximity, convergence and continuity. This framework unifies apparently heterogeneous situations (points of  $\mathbb{R}^n$ , continuous functions, bounded sequences) and opens the way to the topological constructions that will organize the present volume.

---

*The first exercise checks a nonstandard metric. The second compares the familiar balls of the standard norms on  $\mathbb{R}^2$ .*

- Verify that  $d(x, y) = |x^3 - y^3|$  is a metric on  $\mathbb{R}$ . Does it define the same topology as the usual metric?

Solution: Positivity and symmetry are immediate. If  $d(x, y) = 0$ , then  $x^3 = y^3$ , hence  $x = y$  because  $t \mapsto t^3$  is injective. For the triangle inequality, let  $x, y, z \in \mathbb{R}$ . Then  $|x^3 - y^3| = |(x^3 - z^3) + (z^3 - y^3)| \leq |x^3 - z^3| + |z^3 - y^3| = |x^3 - z^3| + |z^3 - y^3|$ . Thus  $d$  is a metric. It defines the same topology as the usual metric because  $t \mapsto t^3$  is a homeomorphism of  $\mathbb{R}$  onto itself. It is not globally Lipschitz-equivalent to the usual metric: for  $x \neq y$ ,  $|x^3 - y^3| = |x - y| |x^2 + xy + y^2|$ . For  $x, y$  near 0,  $|x^2 + xy + y^2|$  is small, so  $|x^3 - y^3|$  is much smaller than  $|x - y|$ . For  $x, y$  far from 0,  $|x^2 + xy + y^2|$  is large, so  $|x^3 - y^3|$  is much larger than  $|x - y|$ . This distinction matters for completeness and uniform continuity. bllpschitz equivalence gives global comparison constants. This distinction matters for completeness and uniform continuity. but it vanishes at  $(0, 0)$ , so no positive lower bound is available there. Topological equivalence preserves the open sets, which tends to 0 near the origin and becomes arbitrarily large far from the origin. On  $[-A, A]$ , the factor is at most  $3A^2$ , which tends to 0 near the origin and becomes arbitrarily large far from the origin. On  $[-A, A]$ , the factor is at most  $3A^2$ .

$$|x^3 - y^3| = |x - y| |x^2 + xy + y^2|$$

Thus  $d$  is a metric. It defines the same topology as the usual metric because  $t \mapsto t^3$  is a homeomorphism of  $\mathbb{R}$  onto itself. It is not globally Lipschitz-equivalent to the usual metric: for  $x \neq y$ ,  $|x^3 - y^3| = |x - y| |x^2 + xy + y^2|$ . For  $x, y$  near 0,  $|x^2 + xy + y^2|$  is small, so  $|x^3 - y^3|$  is much smaller than  $|x - y|$ . For  $x, y$  far from 0,  $|x^2 + xy + y^2|$  is large, so  $|x^3 - y^3|$  is much larger than  $|x - y|$ . This distinction matters for completeness and uniform continuity. bllpschitz equivalence gives global comparison constants. This distinction matters for completeness and uniform continuity. but it vanishes at  $(0, 0)$ , so no positive lower bound is available there. Topological equivalence preserves the open sets, which tends to 0 near the origin and becomes arbitrarily large far from the origin. On  $[-A, A]$ , the factor is at most  $3A^2$ .

$$|x^3 - z^3| + |z^3 - y^3| \leq |x^3 - y^3|$$

Solution: Positivity and symmetry are immediate. If  $d(x, y) = 0$ , then  $x^3 = y^3$ , hence  $x = y$  because  $t \mapsto t^3$  is injective. For the triangle inequality, let  $x, y, z \in \mathbb{R}$ . Then  $|x^3 - y^3| = |(x^3 - z^3) + (z^3 - y^3)| \leq |x^3 - z^3| + |z^3 - y^3| = |x^3 - z^3| + |z^3 - y^3|$ . Thus  $d$  is a metric. It defines the same topology as the usual metric because  $t \mapsto t^3$  is a homeomorphism of  $\mathbb{R}$  onto itself. It is not globally Lipschitz-equivalent to the usual metric: for  $x \neq y$ ,  $|x^3 - y^3| = |x - y| |x^2 + xy + y^2|$ . For  $x, y$  near 0,  $|x^2 + xy + y^2|$  is small, so  $|x^3 - y^3|$  is much smaller than  $|x - y|$ . For  $x, y$  far from 0,  $|x^2 + xy + y^2|$  is large, so  $|x^3 - y^3|$  is much larger than  $|x - y|$ . This distinction matters for completeness and uniform continuity. bllpschitz equivalence gives global comparison constants. This distinction matters for completeness and uniform continuity. but it vanishes at  $(0, 0)$ , so no positive lower bound is available there. Topological equivalence preserves the open sets, which tends to 0 near the origin and becomes arbitrarily large far from the origin. On  $[-A, A]$ , the factor is at most  $3A^2$ .

- Describe the unit balls in  $\mathbb{R}^2$  for  $\|x\|_1$ ,  $\|x\|_2$ , and  $\|x\|_\infty$ .

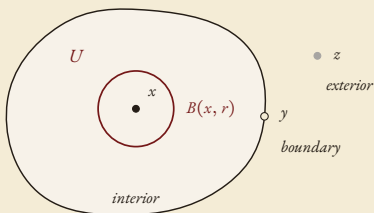
Solution: The Euclidean unit ball is the usual disk. The  $\ell^1$ -unit ball is a diamond with vertices  $(\pm 1, 0)$  and  $(0, \pm 1)$ . The  $\ell^\infty$ -unit ball is  $\{(x, y) : |x| \leq 1, |y| \leq 1\}$ . The inclusions  $B_1(0) \subset B_2(0) \subset B_\infty(0)$  reflect the inequalities  $\|x\|_1 \leq \|x\|_2 \leq \|x\|_\infty$ . In  $\mathbb{R}^n$ ,  $\|x\|_1 \leq \sqrt{n} \|x\|_2 \leq \sqrt{n} \|x\|_\infty$  and  $\|x\|_2 \leq \|x\|_1 \leq \sqrt{2} \|x\|_2$ . All norms are equivalent in finite dimension; this fails in infinite dimension, for instance among function-space norms.

## 2. OPEN SETS, CLOSED SETS, NEIGHBORHOODS.

A metric space has its distance, but one almost never works directly with it. What matters is the geography it draws: which sets leave each of their points some room to maneuver, which sets retain their limits, which sets envelop a point without themselves being open. The notions of open set, closed set and neighborhood, familiar from  $\mathbb{R}^n$ , carry over word for word to metric spaces, but their scope broadens considerably: they organize function spaces and discrete spaces just as well.

**DEFINITION 2.1:** Let  $(X, d)$  be a metric space. A subset  $U \subset X$  is *open* if for every  $x \in U$ , there exists  $r > 0$  such that  $B(x, r) \subset U$ .

The following figure illustrates three situations: an interior point, around which an entire ball fits inside the set; a boundary point, on the edge; and an exterior point.



*Interior point (the ball  $B(x, r)$  is contained in  $U$ ), boundary point, and exterior point.*

Open sets satisfy the fundamental properties of a topology:  $\emptyset$  and  $X$  are open; any union of open sets is open; any finite intersection of open sets is open. The restriction to finite intersections is no artifice: in  $\mathbb{R}$ ,  $\bigcap_{n \geq 1} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ , which is not open. Infinity, even countable infinity, can close what finiteness left open.

**DEFINITION 2.2:** A subset  $F \subset X$  is *closed* if its complement  $X \setminus F$  is open.

The definition by complement is elegant but unwieldy. In practice one prefers the sequential characterization:  $F$  is closed if and only if every sequence of elements of  $F$  that converges in  $X$  has its limit in  $F$ . A closed set is one that does not let its limits escape.

**DEFINITION 2.3:** A set  $V$  is a *neighborhood* of a point  $x$  if there exists an open set  $U$  such that  $x \in U \subset V$ .

**DEFINITION 2.4:** Let  $A \subset X$ . The *interior*  $\overset{\circ}{A}$  is the largest open set contained in  $A$ . The *closure*  $\bar{A}$  is the smallest closed set containing  $A$ . The *boundary* is  $\partial A = \bar{A} \setminus \overset{\circ}{A}$ .

**PROPOSITION 2.5:** A point  $x$  belongs to  $\bar{A}$  if and only if there exists a sequence  $(x_n)$  of elements of  $A$  such that  $x_n \rightarrow x$ .

This sequential characterization of closure is one of the most widely used tools in metric spaces. It converts a set-theoretic question, does  $x$  lie in the smallest closed set containing  $A$ ?, into a constructive procedure: find a sequence of elements of  $A$  converging to  $x$ .

**EXAMPLE:** In  $(\mathbb{R}, |\cdot|)$ , the interior of  $\mathbb{Q}$  is empty ( $\overset{\circ}{\mathbb{Q}} = \emptyset$ ), since every open interval contains irrationals. The closure of  $\mathbb{Q}$  is all of  $\mathbb{R}$ , by density of the rationals. The boundary is therefore  $\partial\mathbb{Q} = \mathbb{R}$ : the rationals are everywhere and nowhere at once. In  $(C([0, 1]), d_\infty)$ , the open ball of center 0 and radius 1 is the set of continuous functions  $f$  satisfying  $\sup_{[0,1]} |f| < 1$ .

**DEFINITION 2.6:** A subset  $A$  is *dense* in  $X$  if  $\bar{A} = X$ , that is, if every point of  $X$  is the limit of a sequence of elements of  $A$ .

**REMARK 2.7:** The sequential characterization of closed sets and closure is specific to metric spaces (and, more generally, to spaces with a countable neighborhood base). In a general topological space, sequences no longer suffice and one must resort to filters or nets. This is one of the privileges of metric spaces: sequences form a complete language for describing the topology.

The topology of a metric space (open sets, closed sets, interior, closure, boundary) is built entirely from open balls, and the sequential characterization makes these notions accessible to concrete computation. This architecture, inherited from  $\mathbb{R}^n$  but valid far beyond it, provides the vocabulary in which completeness, compactness and continuity will be expressed.

---

*The first exercise proves closedness by sequences. The second computes closure, interior, and boundary in a dense example.*

- Show, using the sequential characterization, that

$$F = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

is closed.

Solution: Let  $(x_n, y_n) \in F$  and suppose  $(x, y) \in \mathbb{R}^2$ . Then  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Since  $(x_n, y_n) \in F$  and  $F$  is closed,  $(x, y) \in F$ . Hence  $(x, y) \in F$ . Therefore every convergent sequence in  $F$  has its limit in  $F$ , and  $F$  is closed. Equivalently,  $F$  is the inverse image of  $(-\infty, 1]$  under a continuous function. These are the two standard routes to closedness: test convergent sequences, or use the inverse image of a closed set. The latter is often faster for sets defined by inequalities.

- Compute the closure, interior, and boundary of  $A = \mathbb{Q} \cap [0, 1]$  in  $\mathbb{R}$ .

Solution: The rationals are dense in  $\mathbb{R}$ , so every point of  $[0, 1]$  is the limit of points of  $A$ . No point outside  $[0, 1]$  can be a limit of  $A$ . Thus  $\overline{A} = [0, 1]$ . The interior is empty, because every nonempty open interval contains irrational points and therefore cannot be contained in  $A$ . Consequently  $\text{int}(A) = \emptyset$ . This example separates topological size from measure:  $A$  is dense in  $[0, 1]$ , but it is countable. It is neither open nor closed, but is Borel:  $\mathbb{Q}$  is a countable union of closed singletons, hence an  $F_\sigma$ , and  $[0, 1]$  is closed. Thus  $\lambda(A) = 0$  while  $\lambda(\overline{A}) = 1$ . Closure and interior measure different features, though  $A \subseteq \overline{A} \subseteq \mathbb{R}^d$  always holds.

3. COMPLETENESS.

SOME spaces have holes. In  $\mathbb{Q}$ , the sequence of decimal approximations to  $\sqrt{2}$  is Cauchy: its terms draw ever closer to one another, yet the limit does not belong to  $\mathbb{Q}$ . The sequence *ought to converge* but the space is not rich enough to harbor its limit. Completeness is precisely the absence of such holes: a complete space contains every limit that its Cauchy sequences promise. This property, which has underpinned all of real analysis since the construction of  $\mathbb{R}$ , carries over to metric spaces with spectacular consequences.

**DEFINITION 3.1:** A sequence  $(x_n)$  in a metric space  $(X, d)$  is a *Cauchy sequence* if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $m, n \geq N$  implies  $d(x_m, x_n) < \varepsilon$ .

Every convergent sequence is Cauchy. Terms that approach a limit a fortiori approach one another. The converse is the decisive question.

**DEFINITION 3.2:** A metric space  $(X, d)$  is *complete* if every Cauchy sequence in  $X$  converges in  $X$ .

In other words, a space is complete if sequences that *ought to converge* actually do: the Cauchy criterion, verifiable without knowing the limit, suffices to guarantee its existence.

**EXAMPLE:**  $(\mathbb{R}^n, d_2)$  is complete. The space  $(C([a, b]), d_\infty)$  is complete: this is the uniform limit theorem, which states that the uniform limit of a sequence of continuous functions is continuous. By contrast,  $(\mathbb{Q}, |\cdot|)$  is not complete.

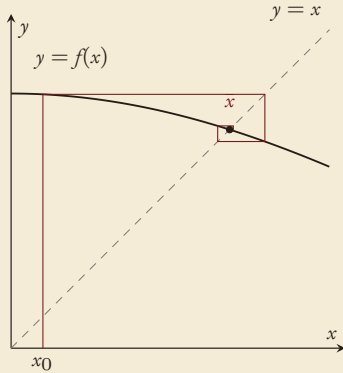
**PROPOSITION 3.3:** A subspace of a complete metric space is complete if and only if it is closed.

This result clarifies the link between topological closure and completeness: being closed in a complete space means inheriting its completeness. To verify that a subset is complete, it suffices to show that it is closed in a complete ambient space.

**THEOREM 3.4: (Banach fixed-point theorem.)** Let  $(X, d)$  be a complete metric space and  $f: X \rightarrow X$  a contraction, that is, a map satisfying  $d(f(x), f(y)) \leq k d(x, y)$  for all  $x, y \in X$ , with  $0 \leq k < 1$ . Then  $f$  has a unique fixed point  $\bar{x} \in X$ , obtained as the limit of any iterated sequence  $x_{n+1} = f(x_n)$ .

**PROOF:** Let  $x_0 \in X$  be arbitrary and  $x_{n+1} = f(x_n)$ . One checks that  $d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$ , whence, by the triangle inequality,  $(x_n)$  is Cauchy. By completeness, it converges to some  $\bar{x}$ . The continuity

of  $f$  (a consequence of the contraction property) gives  $f(\bar{x}) = \bar{x}$ . Uniqueness follows from separation: if  $f(\bar{x}) = \bar{x}$  and  $f(\bar{y}) = \bar{y}$ , then  $d(\bar{x}, \bar{y}) \leq k d(\bar{x}, \bar{y})$ , which forces  $d(\bar{x}, \bar{y}) = 0$ .



*Convergence of the iterated sequence toward the fixed point*

The fixed-point theorem is a remarkable lever: it transforms an existence problem (finding a solution) into a verification problem, showing that an operator is a contraction. Completeness supplies convergence; the contraction supplies uniqueness.

**EXAMPLE:** *The Cauchy-Lipschitz theorem, which guarantees local existence and uniqueness for solutions of ordinary differential equations, is proved by applying Banach's fixed-point theorem in a space of continuous functions equipped with the uniform norm, or an equivalent weighted norm.*

**DEFINITION 3.5:** The *completion* of a metric space  $(X, d)$  is a complete metric space  $(\hat{X}, \hat{d})$  into which  $X$  embeds isometrically and whose image is dense. The completion exists and is unique up to isometry.

**REMARK 3.6:** The construction of the completion generalizes that of  $\mathbb{R}$  from  $\mathbb{Q}$  by equivalence classes of Cauchy sequences. The  $L^p$  spaces, for  $1 \leq p < \infty$ , of chapter 17 arise in the same way: they are completions of spaces of integrable simple functions for the  $L^p$  norms. Filling the holes of a space reveals its deep structure.

Completeness is, in the end, the guarantee that Cauchy sequences converge: it is the contraction that makes the Picard iterates Cauchy, and completeness that then supplies their limit. From Banach's fixed-point theorem to  $L^p$  spaces, every time a Cauchy sequence appears in the sequel, it is the completeness of the ambient space that will close the argument.

*The first exercise exhibits a Cauchy sequence whose limit is missing. The second applies Banach's fixed-point theorem concretely.*

- Construct a Cauchy sequence in  $(\mathbb{Q}, |\cdot|)$  that does not converge in  $\mathbb{Q}$ .

This sequence converges to  $\sqrt{2}$  in  $\mathbb{R}$ , hence it is Cauchy for the usual metric. Since all its terms are rational, it is a Cauchy sequence in  $\mathbb{Q}$ . If it converged in  $\mathbb{Q}$  to some  $q$ , then the same sequence would converge to  $q$  in  $\mathbb{R}$ . By uniqueness of the real limit,  $q = \sqrt{2}$ , impossible in  $\mathbb{Q}$ . Thus  $\mathbb{Q}$  is not complete. For  $m, n \geq N$ , the truncations satisfy  $|x_m - x_n| \leq 10^{-N}$ . Two other rational constructions give the same missing limit. Bisection starts with  $a_0 = 1, b_0 = 2$  and replaces the appropriate endpoint by  $c_n = (a_n + b_n)/2$  according as  $c_n^2 > 2$  or  $c_n^2 < 2$ . Heron-Newton iteration uses the appropriate endpoint by  $c_n = (a_n + 2/a_n)/2$ . Both converge to  $\sqrt{2}$ . This incompleteness is why one constructs  $\mathbb{R}$  as the completion of  $\mathbb{Q}$ ; the construction extends to arbitrary metric spaces.

Solution: Take rational decimal truncations of  $\sqrt{2}$ :

1, 1.4, 1.41, 1.414, 1.4142, ...

• Let  $f : [0, 1] \rightarrow [0, 1]$  be  $f(x) = x/2 + 1/4$ . Verify that  $f$  is a contraction, find its fixed point, and compute the iterates from  $x_0 = 0$ .

In general  $x_n = 1/2 - 2^{-(n+1)}$ , so  $x_n \rightarrow 1/2$  geometrically. This is exactly the convergence predicted by Banach's fixed-point theorem. The self-map condition also holds:  $f([0, 1]) \subseteq [0, 1]$ , a nonempty complete interval. The exact error is  $|x_n - 1/2| = 2^{-(n+1)}$ , equal here to the Banach bound  $k^n |x_0 - 1/2|$  with  $k = 1/2$ . The theorem supplies existence, uniqueness, geometric convergence, and an explicit error estimate.

hence  $x = 1/2$ . Starting from  $x_0 = 0$ ,

$$x_1 = \frac{1}{4}, \quad x_2 = \frac{3}{8}, \quad x_3 = \frac{16}{7}, \quad x_4 = \frac{15}{32},$$

so  $f$  is a contraction with ratio  $1/2$ . Its fixed point satisfies

$$\frac{1}{2} |x - y| = |f(x) - f(y)|,$$

Solution: For  $x, y \in [0, 1]$ ,

4. COMPACTNESS.

WHY do continuous functions on a closed interval always attain their bounds, while this may fail on an open interval? The answer is one word: compactness. In  $\mathbb{R}^n$ , the Heine-Borel theorem identifies compact sets with the closed bounded ones, but this convenient characterization is an accident of finite dimension. As soon as one passes to function spaces, *closed and bounded* no longer suffices, and compactness demands an intrinsic definition.

**DEFINITION 4.1:** A metric space  $(X, d)$  is *sequentially compact* if from every sequence of elements of  $X$  one can extract a subsequence converging in  $X$ .

**DEFINITION 4.2:** A metric space  $(X, d)$  is *compact* (in the open-cover sense) if from every cover of  $X$  by open sets one can extract a finite subcover.

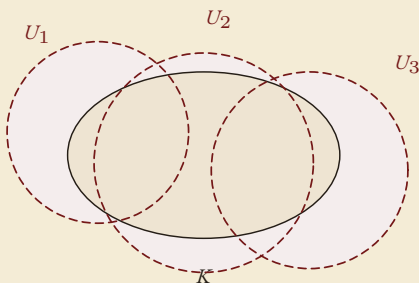
These two definitions appear to address different things (one concerns sequences, the other covers), but in metric spaces they say exactly the same thing.

**THEOREM 4.3:** In a metric space, the following properties are equivalent:

1.  $(X, d)$  is compact (open covers).
2.  $(X, d)$  is sequentially compact.
3.  $(X, d)$  is complete and totally bounded.

*This result is stated without proof.*

A space is totally bounded if, for every  $\varepsilon > 0$ , it can be covered by finitely many balls of radius  $\varepsilon$ . The third characterization decomposes compactness into two ingredients: completeness prevents limits from escaping, and total boundedness prevents sequences from scattering.



*A compact set covered by three open sets: the final state of a finite extraction*

The figure shows the final state: three open sets suffice to cover the compact set. Compactness asserts that from any open cover

(potentially infinite) one can always extract such a finite subfamily: precisely the content of the open-cover definition.

*EXAMPLE:* The interval  $[0, 1]$  is compact in  $\mathbb{R}$ . Every finite metric space is compact. By contrast, the closed unit ball of  $(C([0, 1]), d_\infty)$  is not compact: the sequence  $f_n(x) = x^n$  admits no uniformly convergent subsequence (the pointwise limit is not continuous).

*PROPOSITION 4.4:*

1. Every compact set is closed and bounded.
2. Every closed subset of a compact set is compact.
3. The continuous image of a compact set is compact.

*This result is stated without proof.*

Property (iii) is a transfer principle: compactness, like portable goods, passes from source to target under a continuous map. An immediate consequence: every continuous real-valued function on a compact set is bounded and attains its bounds.

*PROPOSITION 4.5: (Heine's theorem.)* Every continuous map from a compact metric space to a metric space is uniformly continuous: for every  $\varepsilon > 0$  there is  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $d(f(x), f(y)) < \varepsilon$ , the same  $\delta$  serving at every point.

*This result is stated without proof.* The idea of passing from local to uniform requires half-radii: choose  $r_x > 0$  so that  $d(f(x), f(y)) < \varepsilon/2$  whenever  $d(x, y) < r_x$ , then extract a finite cover by the balls  $B(x, r_x/2)$ . The minimum of the chosen half-radii ensures that two sufficiently close points lie in one common ball  $B(x, r_x)$ ; the triangle inequality then bounds the distance between their images by  $\varepsilon$ .

**THEOREM 4.6: (Heine-Borel.)** In  $\mathbb{R}^n$ , a subset is compact if and only if it is closed and bounded. The corresponding sequential form (every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence) is called *Bolzano-Weierstrass*.

*This result is stated without proof.*

This result rests on the completeness of  $\mathbb{R}^n$  and on the fact that in finite dimension, bounded implies totally bounded.

**REMARK 4.7:** In infinite dimension, closed and bounded does not imply compact. The closed unit ball of an infinite-dimensional Banach space (Chapter 3) is never compact for the norm topology: this signals that infinite dimension offers enough room for sequences to escape along ever-new directions uniformly separated from the previous ones (Riesz's lemma) without ever catching up. The Ascoli-Arzelà theorem, which characterizes the *relatively* compact subsets of  $C([a, b])$  by equicontinuity and uniform boundedness (a subset is compact if it is moreover closed), restores an operational criterion where mere boundedness fails.

Compactness turns the infinite into the finite: it guarantees that sequences have convergent subsequences and that covers can be trimmed. This finiteness property will be a decisive lever whenever an argument requires passing from the local to the global.

---

*The first exercise shows why closed and bounded is not enough in infinite dimension. The second uses compactness to prove existence of extrema.*

• In  $E = C([0, 1], \mathbb{R})$  with the sup metric, show that the closed unit ball

$$\overline{B}(0, 1) = \{f : \|f\|_\infty \leq 1\}$$

is not compact.

If a subsequence  $f_{n_k}$  converged uniformly in  $C([0, 1])$ , its uniform limit would be continuous and would also be the pointwise limit above, a contradiction. Hence the closed unit ball is not sequentially compact, and therefore not compact. Closed and bounded does not imply compact in infinite-dimensional normed spaces. Indeed  $\|f_n\|_\infty = 1$ . More generally, Riesz's theorem says that the closed unit ball of any infinite-dimensional Banach space is never norm-compact. In spaces of continuous functions a finer criterion is needed: equicontinuity enters through Ascoli-Arzelà (Exposition 2).

$$f(x) = 0 \leq x < 1, \quad f(1) = 1.$$

Solution: Consider  $f_n(x) = x^n$ . Each  $f_n$  lies in  $\overline{B}(0, 1)$ . Pointwise,  $f_n(x) \rightarrow 0$  for  $0 \leq x < 1$ , while  $f_n(1) = 1$ . The pointwise limit is therefore the discontinuous function

• Let  $K \subset \mathbb{R}^n$  be compact and nonempty and  $f : K \rightarrow \mathbb{R}$  continuous. Prove that  $f$  is bounded and attains its bounds.

This is the extreme value theorem in its metric compactness form. To see why the supremum belongs to the image, approach it by a sequence in  $f(K)$  and use closedness; the infimum is treated identically. This recovers Weierstrass's theorem on a closed interval and also applies to the compact sphere  $S^{n-1}$ . Compactness transfers through continuity and thereby gives existence of extrema.

$$f(a) = \inf_K f, \quad f(b) = \sup_K f.$$

Solution: The continuous image of a compact set is compact, so  $f(K) \subset \mathbb{R}$  is compact. In  $\mathbb{R}$ , compact sets are closed and bounded. Hence  $f(K)$  is bounded, and its supremum and infimum belong to  $f(K)$ . Therefore there exist  $a, b \in K$  such that

5. CONNECTEDNESS.

CAN one break a space into two pieces without tearing anything? If a continuous function on  $[a, b]$  takes a negative value at  $a$  and a positive value at  $b$ , it must vanish somewhere in between: this is the intermediate value theorem. That result, so natural it seems self-evident, rests on a deep property of the interval: it is impossible to split it into two disjoint nonempty open sets. Connectedness formalizes this impossibility of a clean cut.

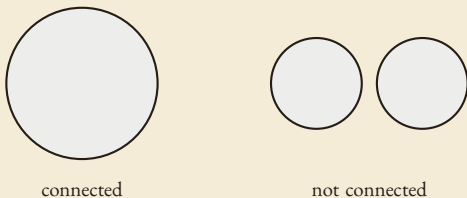
**DEFINITION 5.1:** A metric space  $(X, d)$  is *connected* if the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$ . Equivalently, there is no partition  $X = U \sqcup V$  with  $U$  and  $V$  open and nonempty.

**DEFINITION 5.2:** A metric space  $(X, d)$  is *path-connected* if for all  $x, y \in X$ , there exists a continuous path  $\gamma: [0, 1] \rightarrow X$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ .

**PROPOSITION 5.3:** Every path-connected space is connected. The converse is false in general, but it holds for open subsets of  $\mathbb{R}^n$ . The classical counterexample is the topologist's sine curve, the closure in  $\mathbb{R}^2$  of the graph of  $x \mapsto \sin(1/x)$  on  $(0, 1]$ : it is connected, but no path joins the graph to the limit segment  $\{0\} \times [-1, 1]$ .

*This result is stated without proof.*

Path-connectedness is the constructive version of connectedness: it does not merely assert that the space is in one piece but supplies a path joining any two points. For open subsets of  $\mathbb{R}^n$ , the two notions coincide, a valuable convenience in practice.



*Connected and disconnected spaces in  $\mathbb{R}^2$*

The figure contrasts the two situations: a disk, all in one piece, is connected, while the union of two disjoint disks is not, since each constitutes a nonempty open set that separates the whole into two parts.

**EXAMPLE:**  $\mathbb{R}^n$  and every open ball of  $\mathbb{R}^n$  are path-connected. The space  $\mathbb{Q}$ , with the usual distance, is not connected: for any irrational  $\alpha$ , the partition  $\mathbb{Q} = \{q \in \mathbb{Q} : q < \alpha\} \sqcup \{q \in \mathbb{Q} : q > \alpha\}$  separates  $\mathbb{Q}$  into two nonempty open sets. The deleted comb, the union in  $\mathbb{R}^2$  of the segments  $\{1/n\} \times [0, 1]$ , the base  $[0, 1] \times \{0\}$  and the single point  $(0, 1)$ , is connected but not path-connected: no continuous path joins  $(0, 1)$  to the rest, for its first coordinate would be continuous, hence with connected image, yet near  $(0, 1)$  it would have to take values in the totally disconnected set  $\{0\} \cup \{1/n : n \geq 1\}$  and so remain constant equal to 0 near every instant at which the path passes through  $(0, 1)$ : the set of such instants is thus open, and being also closed (continuity) and nonempty, it is the whole interval: the path is constant.

**DEFINITION 5.4:** The *connected component* of a point  $x \in X$  is the largest connected subset of  $X$  containing  $x$ . The connected components form a partition of  $X$  into closed sets.

**THEOREM 5.5: (Generalized intermediate value theorem.)**

If  $f: X \rightarrow \mathbb{R}$  is continuous and  $X$  is connected, then  $f(X)$  is an interval of  $\mathbb{R}$ .

*This result is stated without proof.*

This theorem reveals that the classical result on  $[a, b]$  had nothing specific to real intervals: it is the connectedness of the source that forces the image to skip no value. The continuous image of a connected set is connected, and the connected subsets of  $\mathbb{R}$  are exactly the intervals.

**REMARK 5.6:** Connectedness is a topological invariant, preserved by homeomorphism. It provides a formidably effective negative tool: to show that two spaces are not homeomorphic, it suffices to find an operation that breaks the connectedness of one but not the other. For instance,  $\mathbb{R}$  with a point removed splits into two pieces,

whereas  $\mathbb{R}^2$  with a point removed remains connected, proving that  $\mathbb{R}$  and  $\mathbb{R}^2$  live in different topological worlds.

With completeness, compactness and connectedness, a metric space now possesses its three fundamental global properties. They form the bedrock on which the existence, uniqueness and extension theorems that pervade all of analysis will rest.

---

*The first exercise distinguishes connected and disconnected planar sets. The second recasts the intermediate value theorem through connectedness.*

- *Decide whether the following subsets of  $\mathbb{R}^2$  are connected: the open disk, the annulus  $1 < x^2 + y^2 < 4$ , the punctured plane  $\mathbb{R}^2 \setminus \{0\}$ , and the plane with the horizontal axis removed.*

Solution: (a) The disk is convex: for  $A, B \in D$ , the path  $\gamma(t) = (1-t)A + tB$  stays in  $D$  by the triangle inequality. Hence it is path-connected. (b) In the annulus, follow a radius from  $A$  to the circle  $x^2 + y^2 = 5/2$ , then a circular arc, then a radius to  $B$ . The path stays in the annulus. The annulus is not convex: the segment joining  $A = (3/2, 0)$  and  $B = (0, 3/2)$  passes through the excluded origin. (c) In the punctured plane, choose  $P$  outside both lines  $(OA)$  and  $(OB)$ . The segments  $[A, P]$  and  $[P, B]$  avoid the origin, proving path-connectedness. If both points lie in the same open half-plane, a straight segment already works. (d) The complement of the horizontal axis is  $\{y > 0\} \cup \{y < 0\}$ , a partition into two nonempty disjoint open sets. Each half-plane is convex, so these are exactly its two connected components. Removing a point from the plane does not disconnect it; removing a line does. Convexity implies path-connectedness, which implies connectedness, but the converses fail.

- *Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous with  $f(0) = -1$  and  $f(1) = 2$ . Use connectedness to show that  $f(c) = 0$  for some  $c \in [0, 1]$ .*

Solution: The continuous image of the connected interval  $[0, 1]$  is an interval. Since it contains  $-1$  and  $2$ , it contains  $0$ . An alternative direct proof takes  $A = \{x \in [0, 1] : f(x) \leq 0\}$  and  $c = \sup A$ , which exists because  $0 \in A$  and  $A$  is bounded above. A sequence  $x_n$  in  $A$  approaching  $c$  gives  $f(c) \leq 0$  by continuity. If  $f(c) > 0$ , then  $c < 1$  since  $f(1) = 2$ , and continuity gives  $\delta > 0$  with  $c + \delta \leq 1$  and  $f(c) + \delta \leq 0$  on  $[c, c + \delta/2]$ . Then  $c + \delta/2 \in A$ , contradicting the definition of  $c$ . Thus  $f(c) = 0$ . The direct proof uses completeness of  $\mathbb{R}$  through the supremum; the abstract proof isolates connectedness as the obstruction to a jump. The same intermediate-value principle holds on every connected source space.



**Study 2.** ★★ METHOD: *the sequential characterization of closure*

The sequential characterization of closure of a subset  $A$  of a metric space:  $x \in \bar{A}$  iff there is a sequence  $(x_n) \in A^{\mathbb{N}}$  with  $x_n \rightarrow x$ . To show a point is in the closure, build an explicit sequence; to show it is not, argue that no sequence in  $A$  can approach it.

Prove that  $\overline{\mathbb{Q}^n} = \mathbb{R}^n$ , and that the closure of  $\{(p, q) \in \mathbb{Q}^2 : p^2 + q^2 < 1\}$  is the closed unit disk  $\{(x, y) : x^2 + y^2 \leq 1\}$ .

- (a) Density of  $\mathbb{Q}^n$  in  $\mathbb{R}^n$ .
- (b) The closure of the rational disk.
- (c) The trivial inclusion.
- (d) The nontrivial inclusion.
- (e) Summarize the method.

*Upside-down solution.*

Solution 2: (a) Density of  $\mathbb{Q}^n$ : Every real  $x$  is a limit of rationals, e.g. the decimal truncations  $q_m = \lfloor 10^m x \rfloor / 10^m \rightarrow x$ . For  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , pick rational sequences  $(q_m^1), \dots, (q_m^n)$  such that  $q_m^i \rightarrow x_i$  for each  $i$ . Then  $(q_m) \in \mathbb{Q}^n$  and  $(q_m) \rightarrow x$ . For  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , pick rational sequences  $(q_m^1), \dots, (q_m^n)$  such that  $q_m^i \rightarrow x_i$  for each  $i$ . Then  $(q_m) \in \mathbb{Q}^n$  and  $(q_m) \rightarrow x$ . Hence  $\overline{\mathbb{Q}^n} = \mathbb{R}^n$ . (b) The rational disk. Let  $A = \{(p, q) \in \mathbb{Q}^2 : p^2 + q^2 < 1\}$  and  $D = \{(x, y) : x^2 + y^2 \leq 1\}$ . (c) Trivial inclusion. Since  $A \subset D$  and  $D$  is closed, taking closures gives  $\bar{A} \subseteq \bar{D} = D$ . (d) Nontrivial inclusion. Let  $(x, y) \in D$ , so  $x^2 + y^2 \leq 1$ . If  $x^2 + y^2 < 1$ , then  $(x, y) \in A$ . If  $x^2 + y^2 = 1$ , approach from inside: the points  $(x_k, y_k) = (1 - 1/k, 1/k)$  lie strictly inside the disk, and choosing  $(p_k, q_k) \in A$  with  $(p_k, q_k) \rightarrow (x, y)$ , and by continuity  $p_k^2 + q_k^2 \rightarrow x^2 + y^2 = 1$  for  $k$  large, so  $(p_k, q_k) \in A$ . If  $x^2 + y^2 = 1$ , approach from inside: the points  $(x_k, y_k) = (1 - 1/k, 1/k)$  lie strictly inside the disk, and choosing  $(p_k, q_k) \in A$  with  $(p_k, q_k) \rightarrow (x, y)$ , and by continuity  $p_k^2 + q_k^2 \rightarrow x^2 + y^2 = 1$  for  $k$  large, so  $(p_k, q_k) \in A$ . If  $x^2 + y^2 = 1$ , approach from inside: the points  $(x_k, y_k) = (1 - 1/k, 1/k)$  lie strictly inside the disk, and choosing  $(p_k, q_k) \in A$  with  $(p_k, q_k) \rightarrow (x, y)$ , and by continuity  $p_k^2 + q_k^2 \rightarrow x^2 + y^2 = 1$  for  $k$  large, so  $(p_k, q_k) \in A$ . (e) Method. Use  $x \in A$  iff some sequence of  $A$  converges to  $x$  to place  $x$  in  $\bar{A}$ , build the sequence explicitly, and when  $x$  sits on the boundary of  $A$ , first move slightly inside and then apply density. To exclude  $x$  from  $\bar{A}$ , exhibit an open ball around  $x$  disjoint from  $A$ .

**Study 3.** ★★ METHOD: *Banach's fixed-point theorem and an application*

Banach's fixed-point theorem: in a nonempty complete metric space  $(X, d)$ , every contraction  $f$  has a unique fixed point  $\bar{x}$ , obtained as the limit of any iteration  $x_{n+1} = f(x_n)$ . Method: check completeness of  $X$ ; check the contraction  $d(f(x), f(y)) \leq k d(x, y)$  with  $k < 1$ ; iterate to compute the fixed point.

Prove the Cauchy–Lipschitz theorem for the ODE  $y' = f(x, y)$ , with  $f$  continuous and Lipschitz in  $y$  on a rectangle around  $(x_0, y_0)$ , via Banach's fixed-point theorem.

- (a) Integral reformulation.



*Upside-down solution.*

Solution 4: (a) *Closed*. Whiting  $S^{n-1}$  with  $g(x) = \|x\|$  continuous and  $\{1\}$  closed,  $S^{n-1}$  is a continuous preimage of a closed set, hence closed. Sequentially: if  $(x_k) \subset S^{n-1}$  and  $x_k \rightarrow x$ , continuity of the norm gives  $\|x\| = \lim \|x_k\| = 1$ , so  $x \in S^{n-1}$ . (b) *Bounded*. Every element has norm 1, so  $S^{n-1} \subseteq \bar{B}(0, 1)$ . (c) *Compact*. By Heine–Borel in  $\mathbb{R}^n$ , closed and bounded implies compact, so  $S^{n-1}$  is compact. (d) *Extrema*. For continuous  $f: S^{n-1} \rightarrow \mathbb{R}$ , the image  $f(S^{n-1})$  is compact in  $\mathbb{R}$ , hence closed and bounded, so it contains its supremum and infimum:  $f$  attains its maximum and minimum. For instance the subordinate matrix norm  $\|A\|$  over the sphere yields an eigenvector of a symmetric matrix, the singular vector of  $A$ , and maximizing  $\langle Ax, x \rangle$  over the sphere yields an eigenvector of a symmetric matrix, the variational route to the spectral theorem. (e) *Method*. In finite dimension, show closed (preimage, sequential characterization, definition by non-strict equalities and inequalities), show bounded (inclusion in a ball) and apply Bolzano–Weierstrass. In infinite dimension, closed and bounded does not imply compact; the general criterion is compactness together with total boundedness, while equicontinuity (Arzelà–Ascoli, see Expedition 2) is the specific criterion in  $C(K)$ .

### Study 5. ★★ METHOD: *path-connectedness and the intermediate value theorem*

**Path-connectedness:** *any two points  $x, y \in X$  are joined by a continuous path  $\gamma: [0, 1] \rightarrow X$  with  $\gamma(0) = x, \gamma(1) = y$ . It implies connectedness (the converse fails in general). The generalized intermediate value theorem: if  $f: X \rightarrow \mathbb{R}$  is continuous and  $X$  is connected, then  $f(X)$  is an interval. Method: to obtain an intermediate value theorem, check that the domain is connected.*

*Prove that a convex subset of  $\mathbb{R}^n$  is path-connected (hence connected), and deduce that the intermediate value theorem extends to any continuous  $f: C \rightarrow \mathbb{R}$  with  $C$  convex.*

- (a) *Convex  $\Rightarrow$  path-connected.*
- (b) *Path-connected  $\Rightarrow$  connected.*
- (c) *The generalized intermediate value theorem.*
- (d) *Application.*
- (e) *Summarize the method.*



**METHOD RECAP**

**METRIC:**  $d : X^2 \rightarrow \mathbb{R}_+$  satisfies separation, symmetry, and the triangle inequality.

**EQUIVALENT METRICS:** They define the same open sets. In finite dimension all norms are equivalent.

**OPEN AND CLOSED BALLS:**  $B(x, r) = \{y : d(x, y) < r\}$ ;  $\bar{B}(x, r) = \{y : d(x, y) \leq r\}$ .

**OPEN SET:**  $U$  is open if every  $x \in U$  has some  $r > 0$  with  $B(x, r) \subseteq U$ .

**CLOSED SET:**  $F$  is closed if  $X \setminus F$  is open. A sequence in  $F$  converging in  $X$  has its limit in  $F$ .

**CLOSURE:**  $\bar{A} = \{x \in X : \exists (x_n) \in A^{\mathbb{N}}, x_n \rightarrow x\}$ , the smallest closed set containing  $A$ .

**CAUCHY SEQUENCE:** For every  $\varepsilon > 0$ , some  $N$  satisfies  $m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon$ .

**COMPLETE SPACE:** Every Cauchy sequence converges. Examples:  $\mathbb{R}$ ,  $\mathbb{R}^n$ ,  $C([a, b])$  with  $d_\infty$ .

**BANACH FIXED-POINT THEOREM:** A contraction with ratio  $k < 1$  on a nonempty complete space has a unique fixed point reached by iteration.

**SEQUENTIAL COMPACTNESS:** Every sequence has a convergent subsequence. In metric spaces this is equivalent to every open cover having a finite subcover.

**BOLZANO-WEIERSTRASS:** In  $\mathbb{R}^n$ , compactness is equivalent to being closed and bounded. This fails in infinite dimension.

**CONTINUOUS IMAGE OF A COMPACT SET:** It is compact. A continuous real-valued function on a nonempty compact set attains its extrema.

**CONNECTEDNESS:** No partition  $X = U \sqcup V$  into nonempty open sets. The connected subsets of  $\mathbb{R}$  are its intervals.

**GENERALIZED INTERMEDIATE VALUE THEOREM:** *If  $f : X \rightarrow \mathbb{R}$  is continuous and  $X$  connected, then  $f(X)$  is an interval.*

**BAIRE THEOREM:** *In a complete metric space, a countable intersection of dense open sets is dense; dually, a countable union of closed sets with empty interior has empty interior.*

## EXPEDITIONS

### EXPEDITION 1: THE BAIRE CATEGORY THEOREM

*Prerequisites: studies 6.*

The Baire category theorem (René-Louis Baire, 1899, doctoral thesis) states that a nonempty complete metric space is never a countable union of "small" sets. This seemingly technical property founds three pillars of modern functional analysis: the uniform boundedness principle, the open mapping theorem, and the closed graph theorem. It also proves unexpected results: in the sense of Baire, almost every continuous function is nowhere differentiable, and almost every real number is a Liouville number.

**(a)** *State the theorem (proved in article 21 used in Study 6, and recalled here): in a complete metric space, the intersection of a countable sequence of dense open sets is dense.*

**(b)** *Dual reformulation: a countable union of closed sets with empty interior has empty interior.*

**(c)** *The Banach–Steinhaus theorem (1927): a pointwise bounded family of operators is uniformly bounded.*

**(d)** *Generic existence of rare phenomena: nowhere differentiable functions (Banach, 1931).*

**(e)** *Generic topology: the "most frequent" type is not always the intuitive one.*

*The Baire category theorem is one of the foundations of functional analysis.*

**History.** René-Louis Baire (1874–1932, French mathematician), in his doctoral thesis (*Sur les fonctions de variables réelles*, 1899), introduced the notion of a meager set (in modern form, a countable union of nowhere dense sets) and proved that  $\mathbb{R}$ , like every complete metric space, is not meager in itself. This seemingly technical idea turned out to be a structuring tool for all of twentieth-century analysis.

**Statement.** Let  $(X, d)$  be a complete metric space (or, more generally, locally compact), and let  $(U_n)_{n \geq 1}$  be a sequence of dense open sets of  $X$ ; then their intersection  $\bigcap_n U_n$  remains dense. The dual form follows by passing to complements: if  $(F_n)_{n \geq 1}$  is a sequence of closed sets with empty interior, their union  $\bigcup_n F_n$  still has empty interior. The accompanying vocabulary: a  $G_\delta$  is a countable intersection of open sets, an  $F_\sigma$  a countable union of closed sets, a set is meager (or of first category) when it is a countable union of nowhere dense sets, and comeager or residual when it is the complement of a meager set; in a complete space a comeager set contains a dense  $G_\delta$ , hence is dense. In this compact form, the theorem says that in a complete space the complement of a meager set is dense.

**Applications.** The theorem runs through all of functional analysis. The Banach–Steinhaus theorem, or uniform boundedness principle, is its most direct consequence: for a family  $(T_\alpha)$  of continuous linear operators from a Banach space  $X$  to a normed space  $Y$ , pointwise boundedness ( $\sup_\alpha \|T_\alpha x\| < \infty$  for each  $x$ ) forces uniform boundedness ( $\sup_\alpha \|T_\alpha\| < \infty$ ). The proof sets  $E_n = \{x : \sup_\alpha \|T_\alpha x\| \leq n\}$ : each  $E_n$  is closed and their union is all of  $X$ , so Baire's theorem forbids them all from having empty interior; one of them therefore contains a ball  $\bar{B}(x_0, r)$ , and homogeneity then gives  $\sup_\alpha \|T_\alpha\| \leq 2n_0/r$ . The same mechanism yields the open mapping theorem (Banach, 1929), by which every continuous surjective linear operator between Banach spaces is open, and the closed graph theorem, which makes continuous any linear operator defined on an entire Banach space, with values in another Banach space, whose graph is closed.

Baire's theorem also serves to establish the existence of exotic objects through a genericity argument. Banach showed in 1931 that the set of functions in  $C([0, 1])$  that are nowhere differentiable is comeager (it contains a dense  $G_\delta$ ): almost every continuous function, in the sense of Baire, is differentiable at no point, so that the "regular" class suggested by intuition is in fact the exception. Likewise, the Liouville numbers, those irrationals approximated



and underlies the extraction of convergent subsequences on which the existence theorems of Cauchy–Peano, Cauchy–Lipschitz, and Schauder all rest.

(a) State the theorem:  $\mathcal{F} \subseteq C([a, b])$  is relatively compact (has compact closure) if and only if  $\mathcal{F}$  is uniformly bounded and equicontinuous.

(b) Define equicontinuity:  $\forall \varepsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F}, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ .

(c) Idea of the proof (article 22 gives it in full, it is not repeated here): diagonal extraction on a countable dense subset, then passage to uniform convergence by equicontinuity.

(d) Application: the Cauchy–Peano theorem (existence of ODE solutions without a Lipschitz hypothesis).

(e) Link with compactness in Banach spaces and Schauder’s fixed-point theorem (statement taken for granted, Chapter 13).

The Ascoli–Arzelà theorem characterizes compactness in the space of continuous functions.

**History.** Giulio Ascoli (1843–1896, Italian mathematician) established in 1883–1884 the sufficient condition: equicontinuity together with uniform boundedness guarantees the existence of a uniformly convergent subsequence. Cesare Arzelà (1847–1912) proved the necessary condition in 1895 and gave the complete modern formulation of the theorem.

**Why it is needed.** At the turn of the twentieth century, nascent functional analysis needed compactness criteria valid outside  $\mathbb{R}^n$ . The Bolzano–Weierstrass theorem, which identifies compactness with closed and bounded, no longer works in infinite dimension; Ascoli–Arzelà supplies the missing criterion for  $C([a, b])$ .

**Statement.** Let  $(K, d)$  be a compact metric space, for instance  $K = [a, b]$ . A subset  $\mathcal{F} \subseteq C(K, \mathbb{R})$  is relatively compact (has compact closure in  $(C(K), d_\infty)$ ) if and only if it satisfies the two following conditions: uniform boundedness,  $\sup_{f \in \mathcal{F}} \|f\|_\infty < \infty$ ; and equicontinuity,  $\forall \varepsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F}, \forall x, y \in K, d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ . Equicontinuity is a continuity “uniform in  $f$ ”: the same  $\delta$  works for every function of  $\mathcal{F}$ , which therefore share a common local regularity. The most frequent case is an equi-Lipschitz family,  $|f(x) - f(y)| \leq L|x - y|$  with a common constant  $L$ , which immediately implies equicontinuity.

**Proof.** *It is given in article 22; one retains its two movements, diagonal extraction on a countable dense subset of  $K$ , then promotion of pointwise convergence to uniform convergence by equicontinuity, and, for the converse, the finite cover of the closure by balls, each center being uniformly continuous on the compact  $K$ .*

**Applications.** *The theorem drives many existence results. The Cauchy–Peano theorem (1886) states that, for a right-hand side that is merely continuous, the differential equation  $y' = f(x, y)$ ,  $y(x_0) = y_0$  admits at least one local solution, with no Lipschitz hypothesis and hence no guarantee of uniqueness: one works on a compact rectangle  $[x_0, x_0 + a] \times [y_0 - b, y_0 + b]$  where  $M = \max |f|$  exists, then restricts to  $[x_0, x_0 + b]$  with  $b = \min(a, b/M)$  (any  $b \leq a$  if  $M = 0$ ) so that the Euler polygons of step  $b/n$  stay inside; they then form a uniformly bounded, equicontinuous family (slope bounded by  $M$ ), and Ascoli–Arzelà extracts a uniformly convergent subsequence whose limit satisfies the equation. Schauder’s fixed-point theorem (1930), which extends Brouwer’s to compact convex sets of a Banach space, rests on the same principle of reduction to finite dimension, and leads to existence theorems in partial differential equations and fluid mechanics (the Navier–Stokes equations). The compactness of the unit ball of the Hölder space  $C^{0,\alpha}([a, b])$ ,  $0 < \alpha \leq 1$ , with its full norm in  $C([a, b])$ , another consequence of Ascoli, is a key tool in harmonic analysis and PDE, and the compensated compactness theory of Murat and Tartar (1970s) extends its spirit to nonlinear problems. The theorem has its limits, since it concerns only continuous functions on a compact set; its extensions carry the names of Kolmogorov–Riesz for  $L^p$  spaces and Aubin–Lions for evolution problems.*

**Philosophy.** *Compactness in a function space demands two simultaneous controls: that the functions not be too large (boundedness) and that they not vary too fast (equicontinuity). This second control is what replaces the Bolzano–Weierstrass theorem, which is powerless in infinite dimension.*

*Upside-down report.*

[illegible]

### EXPEDITION 3: CANTOR SET

*Prerequisites: studies 2, 4.*

*The Cantor set (Georg Cantor, 1883) is one of the most paradoxical objects in mathematics. Built by successive removal of middle thirds of intervals, it is at once compact (closed and bounded), totally disconnected (no interval), perfect (every point is a limit of other points), of measure zero yet uncountable (cardinality of the continuum), and self-similar. These seemingly contradictory properties illustrate the unsuspected richness of the topology of subsets of  $\mathbb{R}$ .*

- (a) *Construction:*  $C_0 = [0, 1]$ . At each stage  $n$ , remove the open middle third of every subinterval.  $C = \bigcap_n C_n$ .
- (b) *Characterization:*  $C = \{x \in [0, 1] : x = \sum_{k \geq 1} a_k / 3^k, a_k \in \{0, 2\}\}$  (no digit 1 in the ternary expansion).
- (c) *Compact, totally disconnected, perfect.*
- (d) *Lebesgue measure 0, yet cardinality  $2^{\aleph_0} = \mathfrak{c}$  (the continuum).*
- (e) *Self-similarity (fractal), Hausdorff dimension  $\log 2 / \log 3 \approx 0.631$ .*

*The Cantor set is the archetype of fractal objects and a challenge to intuition.*

**History.** Georg Cantor (1845–1918, German mathematician) introduced this set in 1883 in *Über unendliche, lineare Punktmannigfaltigkeiten* ("On infinite linear point sets"), one of his foundational works in set theory, while searching for counterexamples to naive conjectures about the topology of  $\mathbb{R}$ .

**Construction.** Start from  $C_0 = [0, 1]$  and remove at each stage the open middle third of every remaining interval: the first stage removes  $(1/3, 2/3)$  and leaves  $C_1 = [0, 1/3] \cup [2/3, 1]$ , the second leaves  $C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$ , and in general  $C_n$  is a union of  $2^n$  closed intervals of length  $3^{-n}$ . The Cantor set is the intersection  $C = \bigcap_{n \geq 0} C_n$ . It admits a convenient arithmetic description: every  $x \in [0, 1]$  can be written  $x = \sum_{k \geq 1} a_k 3^{-k}$  with  $a_k \in \{0, 1, 2\}$ , and removing the middle thirds amounts exactly to forbidding the digit 1, so that  $C = \{x : a_k \in \{0, 2\} \text{ for every } k\}$ . One need only resolve the ambiguity of ternary expansions, for instance by writing  $1/3 = 0.0222 \dots$  rather than  $0.1000 \dots$ ; this way  $1/3$  and  $2/3$  both belong to  $C$ .

**Paradoxical properties.** The set  $C$  combines properties that seem incompatible. It is compact, being a closed (intersection of the closed sets  $C_n$ ) and bounded subset of  $\mathbb{R}$ . It is totally disconnected, meaning it contains no interval: a nontrivial interval  $[a, b]$  is too long to fit inside a single piece of length  $3^{-n}$  once  $n$  is large enough, so it must meet a removed third. Yet it is perfect, that is, closed with no isolated point: by altering the ternary expansion of a point  $x \in C$  from some sufficiently large rank onward, one produces points of  $C$  distinct from  $x$  that converge to it. It has Lebesgue measure zero, since  $\lambda(C) \leq \lambda(C_n) = (2/3)^n$  for every  $n$ . And despite this zero length it has the cardinality of the continuum: the correspondence  $a_k \leftrightarrow a_k/2$  gives a bijection between  $C$  and  $\{0, 1\}^{\mathbb{N}}$ , hence a surjection onto  $[0, 1]$  via binary expansion (the countable ambiguity at dyadic points does not affect cardinality, by a double injection), so  $|C| = 2^{\aleph_0} = \mathfrak{c}$  and  $C$  has as many points as the whole segment.

**Fractal structure.** The set is self-similar:  $C = \frac{1}{3}C \cup (\frac{1}{3}C + \frac{2}{3})$  is the union of two copies of itself scaled by a factor  $1/3$ . The dimension theorem under the open set condition (admitted here) gives  $d = \log N / \log(1/r)$  for  $N$  similarities of the same ratio  $0 < r < 1$ . Here the images of  $(0, 1)$  are disjoint and contained in  $(0, 1)$ , verifying the condition and giving a fractional Hausdorff dimension  $d_H = \log 2 / \log 3 \approx 0.631$ , between that of a point and that of an interval. Topologically,  $C$  is homeomorphic to

*the abstract Cantor space  $\{0, 1\}^{\mathbb{N}}$  with the product topology, and a theorem of Brouwer (1910) supplies the converse: every nonempty compact, totally disconnected, perfect metric space is homeomorphic to it, making  $C$  a universal model. Varying the lengths removed (taking away  $1/4$ , then  $1/16$ , and so on) produces the Smith–Volterra–Cantor set, or “fat Cantor set,” which keeps the same topological structure while having strictly positive measure, for instance  $1/2$ .*

**Applications and legacy.** *The Cantor set runs through several fields. In measure theory, it shows that an uncountable set can be negligible, sharply separating “topologically small” from “small in measure.” In set theory, it accompanies Cantor’s introduction of infinite cardinals and his proof that  $|\mathbb{R}| > |\mathbb{N}|$  (1874, later by the diagonal argument in 1891). In topology and dynamics, the abstract Cantor space reappears everywhere, from  $p$ -adic spaces to symbolic dynamics and coding theory. It is finally the prototype of the fractal attractor, on which Mandelbrot drew in founding the theory of fractals (1975). History also records that Cantor faced the hostility of Kronecker, to whom is attributed, via Weber’s 1893 memorial address, our only source for the remark, the saying “God made the integers, all the rest is the work of man”; Cantor’s ideas on the actual infinite are today counted among the most fertile sources of modern mathematics.*

Upside-down report.

Report 3: (a) *Iterative construction.*  $C_0 = [0, 1]$ .  $C_n$  obtained from  $C_{n-1}$  by removing the open middle third of each interval making up  $C_{n-1}$ . Structure of  $C_n$ :  $2^n$  closed intervals of length  $3^{-n}$ . Cantor set:  $C = \bigcup_{n \in \mathbb{N}} C_n$ . (b) *Ternary characterization.* Theorem:  $C = \{x \in [0, 1] : x \text{ has a ternary representation with } a_k \in \{0, 2\} \forall k\}$ . The  $2^n$  closed intervals of  $C_n$  correspond to prefixes of length  $n$  over  $\{0, 2\}$ . For  $x \in C$ , the nested intervals determine digits and their lengths  $3^{-n}$  tend to zero, giving  $x = \sum_{k \in \mathbb{N}} a_k 3^{-k}$ . Conversely, such a sum lies in every  $C_n$ . Existence of a representation without 1 is required, not the property for every representation:  $1/3 = 0.02222\ldots = 0.1000\ldots$ . (c) *Compact.*  $C$  is bounded in  $[0, 1]$  and closed (intersection of closed sets); by Bolzano-Weierstrass in  $\mathbb{R}$ ,  $C$  is compact. (d) *Totally disconnected.* No interval  $\emptyset \subsetneq [a, b] \subseteq C$  with  $a < b$ , then  $b - a > 3^{-n}$  for  $n$  large enough, but  $C_n$  is a union of intervals of length  $3^{-n}$ , so  $[a, b]$  cannot fit in a single piece, contradicting  $[a, b] \subseteq C$  (c) *Perfect.* For  $x \in C$  there is  $(x_n) \in C \setminus \{x\}$  with  $x_n \rightarrow x$ : write  $x = 0.a_1a_2a_3\ldots$  with  $a_k \in \{0, 2\}$ , define  $x_n$  with the same expansion as  $x$  up to rank  $n$ , then flip digit  $n + 1$  (0 to 2 or 2 to 0); then  $x_n \in C$ ,  $x_n \neq x$ ,  $|x - x_n| \leq 2/3^{n+1} \rightarrow 0$ , so  $x$  is a limit of distinct elements of  $C$  and is not isolated. (f) *Lebesgue measure.*  $0. \lambda(C_n) = 2^n \cdot 3^{-n} = (2/3)^n$ , since  $C \subseteq C_n$ ,  $\lambda(C) \leq (2/3)^n$  for every  $n$ , hence  $\lambda(C) = 0$ . (g) *Cardinality.*  $|C| = |\{0, 2\}^{\mathbb{N}}| = 2^{\aleph_0} = c$ .  $C$  has the cardinality of the continuum, hence is uncountable: an apparent paradox, a measure-zero set with as many points as  $[0, 1]$ . (h) *Self-similarity and fractal dimension.*  $C = \alpha(C) \cup (\alpha(C) + 2/3)$  with  $\alpha(x) = x/3$ ; the Cantor set is the union of 2 copies of itself, each  $1/3$  as large. Under the open set condition, verified here (theorem admitted), for  $N$  similarities of common ratio  $0 < r < 1$ , the Hausdorff dimension is  $d = \log N / \log(1/r) = \log 2 / \log 3 \approx 0.6309$ , strictly between 0 and 1. (i) *Variant: the fat Cantor set.* Instead of removing  $1/3$ , remove  $1/4$ ,  $1/16$ ,  $1/64$ ,  $\ldots$  at successive stages; the total measure removed is  $\sum_{n \in \mathbb{N}} 2^{-n-1}/4^n = 1/2$ , so the fat Cantor set has measure  $1 - 1/2 = 1/2$ , yet is structurally like the ordinary Cantor set: compact, totally disconnected, perfect. This shows that positive measure and total disconnectedness can coexist. (j) *Abstract Cantor space.*  $\{0, 1\}^{\mathbb{N}}$  with the product topology, metrized by  $d(x, y) = 2^{-\min\{k: x_k \neq y_k\}}$  for  $x \neq y$  and  $d(x, x) = 0$ , is compact, totally disconnected, and perfect. Brouwer's theorem (1910, admitted here): every nonempty compact, totally disconnected metric space with no isolated point is homeomorphic to the abstract Cantor space, hence to  $C$ , a universal characterization. (k) *Method.* The Cantor set in brief: construction by iterated removal, a simple ternary characterization, paradoxical properties (compact + totally disconnected + perfect + measure zero + cardinality of the continuum + fractal). It appears throughout mathematics: set theory, measure theory, topology, dynamics, coding theory, fractals: the universal counterexample.

FURTHER READING

- *Fréchet, M. (1906), thesis introducing metric spaces.*
- *Kelley, J. L. (1955), General Topology, Van Nostrand.*
- *Munkres, J. R. (2000), Topology, Prentice Hall.*
- *Willard, S. (1970), General Topology, Addison-Wesley.*