



Articulated Course
in
Mathematics
II
Foundations

FROM STATEMENT TO SPACE

William de France

CLEMENTINIUM EDITIONS

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FOUNDATIONS

A mathematician's patterns, like the painter's or the poet's, must be beautiful.

G. H. HARDY

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Collection ARTICULATED COURSE IN MATHEMATICS

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FOREWORD

This volume builds the foundations of university mathematics in two hundred and ninety-one articles. It accompanies the reader from the first steps in the language of definitions and proofs (where words must be held to account, and each term must carry exactly the same meaning wherever it appears) to the threshold of graduate mathematics. The next volume will develop functional analysis, commutative algebra, and measure theory.

The series Articulated Course in Mathematics is built on an intellectual thesis that runs through all five volumes, from Elements to Mathematical Physics: mathematics is the generative grammar by which reality composes itself. Not a tool forged to describe nature, but the internal syntax of what is. If the Elements volume taught the first words (number, distance, angle, equation), this one teaches the grammar: the structures that govern those words and the rules by which they combine. Order on the real line founds analysis; the group distills into three axioms the common pattern behind the addition of integers, the composition of permutations, and the multiplication of matrices; the vector space unifies $\mathbb{R}^n$, polynomials, and continuous functions under a single axiomatic roof; the metric space abstracts the notion of proximity and carries it to objects that $\mathbb{R}^n$ does not contain.

The architecture of this book is that of the entire series: the fundamental unit is the article, not the chapter. Each article treats a single idea, in two hundred and fifty to eight hundred words. It opens on a tension or a surprise, never on a bare definition. It closes on a sentence that sounds a deeper principle than its immediate content. Articles are gathered into chapters and parts. The equivalence classes of set theory illuminate the quotient $\mathbb{Z}/n\mathbb{Z}$ of arithmetic; Euclidean division of integers illuminates that of polynomials. The same principles (duality, symmetry, passage from local to global, approximation, linearization, quotient, functoriality) reappear in different guises at every turn.

Reading is the first gesture. But learning the grammar of mathematics requires practicing it, and this volume accompanies every chapter with

a program of exercises designed so that the reader is never left stranded. After each article, short practice exercises put the idea just presented into immediate practice. At the end of each chapter, method studies teach the fundamental techniques: solving a quadratic equation with complex coefficients, basis and dimension of a subspace, Grassmann's formula, the rank theorem, complete study of a curve, arc length, the integral test, the Gaussian integral in polar coordinates, Newton's method, classical inequalities. Each is introduced by a lesson, then applied step by step. A summary gathers these methods onto a single page. Finally, guided expeditions lead the reader, in stages, toward a result they would not have believed within reach: the brachistochrone, the isoperimetric inequality, the Steinitz exchange lemma, roots of unity as regular polygons, the complex exponential and trigonometric formulas, the fundamental theorem of plane curves. Every solution is given, detailed and generous, printed upside down at the foot of each exercise: the reader can attempt the exercise without seeing the answer, then consult it by turning the book over.

Each chapter closes with a Further Reading box pointing to the canonical literature for what comes next: Rudin, Apostol, Spivak, Strang, Axler in English; Liret–Martinai, Gourdon, Pommellet in French; a few historical classics when illuminating. Half a page of brief annotations: these are not exhaustive bibliographies but signposts toward the natural continuation.

Ten companion objects run through the whole collection like guiding threads (cf. "To the Reader"): the circle S^1 , the integers $\mathbb{Z}$, the extension $\mathbb{Q}(\sqrt{2})$, the elliptic curve $y^2 = x^3 - x$, the group GL_2 , the function $\zeta(s)$, the space L^2 , the ring $K[X]$, the symmetric group $\mathfrak{S}_3$, and the torus $\mathbb{T}^2$. The undergraduate cycle treats several of them explicitly: $\mathbb{Z}$ reveals itself to be a Euclidean domain, a principal ideal domain, and a unique factorization domain; S^1 becomes the group of complex numbers of modulus 1; $\mathfrak{S}_3$ appears as the smallest non-abelian group; $K[X]$ receives Euclidean division and becomes a model principal ideal domain. The elliptic curve, the group GL_2 , the function $\zeta(s)$, the extension $\mathbb{Q}(\sqrt{2})$, the space L^2 , and the torus $\mathbb{T}^2$ appear here only in outline and will be fully developed in later volumes. Alongside these ten canonical objects, certain pedagogical companions recurring through the undergraduate cycle (the space $\mathbb{R}^2$, the polynomial $X^2 + 1$, the exponential, the prime numbers, the finite field $\mathbb{F}_p$) structure the progression of the chapters and will be frequently called on as examples.

The final parts of the volume open onto Euclidean geometry, combinatorics and probability, multiple integration and numerical computation: the base on which graduate mathematics will build.

The volume assumes mastery of secondary-school content: algebraic manipulation, standard functions, plane and solid geometry, rudiments of probability. It prepares the reader for the next volume, Graduate Mathematics, which develops graduate topics: functional analysis, Galois theory, differential geometry, distributions, and the tools of transition toward research.

The notation is standard. It follows the French convention: $\mathbb{N}$ includes zero, $\subset$ means inclusion allowing equality, $f: A \rightarrow B$ denotes a map, $\|\cdot\|$ a norm. Each new notation is defined at its first occurrence.

The student who opens this volume is about to pass from manipulating words to understanding the rules that govern them. It is a threshold. On the other side, mathematics is no longer a collection of recipes: it is an edifice whose plan one begins to see.

TO THE READER

This course articulates in five volumes the grammar by which reality composes itself: over thirteen hundred articles of mathematics, from the first whole number to the Langlands correspondence and from the brachistochrone to the amplitubedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.

Each article develops a single idea, from its motivation to its interpretation. Definitions are framed. Results (propositions, theorems) are marked by a bold label; in volumes 2 and 3, major theorems are further framed in red. Proofs are present where they illuminate, sketched where full rigor would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The text distinguishes these three cases.

Ten mathematical objects run through the collection as recurring examples: the circle S^1 , the integers $\mathbb{Z}$, the extension $\mathbb{Q}(\sqrt{2})$, the elliptic curve $y^2 = x^3 - x$, the group GL_2 , the function $\zeta(s)$, the space L^2 , the ring $K[X]$, the symmetric group $\mathfrak{S}_3$, and the torus T^2 . From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.

The figures, numbering more than eight hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colors. They do not replace proof; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.

In volumes I–III, each chapter extends into graded exercises (practice exercises, method studies, expeditions), whose detailed solutions are printed upside down at the foot of each exercise: the reader may attempt the exercise without seeing the answer, then read it by turning the book around. Whoever engages with them will find a teacher, never an examiner.

The reader may follow the linear path or take the cross-volume passages. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands program; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.

Further reading points to English editions where these can be identified, or to verified English alternatives with the same purpose. Historical attributions are retained; works available only in another language are identified as such. These readings extend the course rather than form part of its prerequisites.

Every error reported is an error corrected.

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NOTE ON THE LEVELS

The five volumes follow a progression from the first mathematical objects to questions of research:

- Elements (I): *secondary-school mathematics, from arithmetic to the first proofs.*
- Foundations (II): *the foundations of university mathematics.*
- Graduate Mathematics (III): *deeper study and preparation for research.*
- Correspondences (IV): *the research level, from class field theory to the Langlands program.*
- Mathematical Physics (V): *the research level, from symmetries to strings.*

These levels guide the reader without imposing an academic timetable. Parts and chapters mark the stages of the journey; the pace depends on the reader's background and the questions they wish to explore.

Volumes I, II, and III successively build the prerequisites for the next volume. Readers may enter at the level for which they have the necessary background. Volumes IV and V share the foundation of Graduate Mathematics and may be read independently of each other. The bridges at the beginning of each volume explain the change in perspective.

NOTATIONS

$\mathbb{N}$	<i>natural numbers, including zero</i>
$\mathbb{Z}$	<i>integers</i>
$\mathbb{Q}$	<i>rational numbers</i>
$\mathbb{R}$	<i>real numbers</i>
$\mathbb{C}$	<i>complex numbers</i>
$\in$	<i>belongs to</i>
$\subset$	<i>contained in, allowing equality</i>
$\cup, \cap, \emptyset$	<i>union, intersection, empty set</i>
$\mathcal{P}(E)$	<i>power set of E</i>
$A \setminus B, A^c$	<i>set difference, complement</i>
$\forall, \exists$	<i>for all, there exists</i>
$\Rightarrow, \Leftrightarrow$	<i>implies, is equivalent to</i>
$ x $	<i>absolute value of x</i>
$\sqrt{x}$	<i>square root of x</i>
$\leq, \geq$	<i>less than or equal to, greater than or equal to</i>
π	<i>ratio of circumference to diameter</i>
$\lfloor x \rfloor$	<i>floor of x</i>
∞	<i>infinity</i>
$\sin, \cos, \tan, \arctan$	<i>trigonometric functions and inverse</i>
$\ln, \exp$	<i>natural logarithm, exponential</i>
e	<i>Euler's number ($e \approx 2.718$)</i>
$f'(x)$	<i>derivative of f at x</i>
$\int_a^b f(x) \, dx$	<i>integral of f from a to b</i>
$\sum, \prod$	<i>sum, product</i>
$n!, \binom{n}{k}$	<i>factorial, binomial coefficient</i>
i	<i>imaginary unit, $i^2 = -1$</i>
$\bar{z}, z , \arg(z)$	<i>conjugate, modulus, argument</i>
$\operatorname{Re}(z), \operatorname{Im}(z)$	<i>real part, imaginary part</i>
$\rightarrow$	<i>tends to</i>
$x \mapsto f(x)$	<i>the map sending x to $f(x)$</i>
$\vec{u}$	<i>vector u</i>
$f \circ g$	<i>composition of f and g</i>

$\ x\ $	<i>norm of x</i>
$\langle x, y \rangle$	<i>inner product of x and y</i>
$\dim V$	<i>dimension of a vector space V</i>
$\ker f, \operatorname{Im} f$	<i>kernel, image of a linear map</i>
$\det A$	<i>determinant of a matrix A</i>
$A^T, M_n(\mathbb{R})$	<i>transpose, square matrices of order n</i>
$\operatorname{Vect}(v_1, \dots), \operatorname{rg} f$	<i>spanned subspace, rank</i>
$\operatorname{GL}_n, \operatorname{SL}_n, \operatorname{O}_n$	<i>general linear, special linear, orthogonal groups</i>
$\oplus, \otimes$	<i>direct sum, tensor product</i>
$\lim$	<i>limit</i>
$\sup, \inf$	<i>supremum, infimum</i>
$\limsup, \liminf$	<i>limit superior, limit inferior</i>
$O(\cdot), o(\cdot), \sim$	<i>Landau notation, asymptotic equivalence</i>
$\frac{d}{dx}, \frac{\partial}{\partial x}$	<i>ordinary derivative, partial derivative</i>
∇f	<i>gradient of f</i>
L^p	<i>Lebesgue space of exponent p</i>
C^k	<i>functions with continuous derivatives through order k</i>
C^∞	<i>infinitely differentiable functions</i>
μ, λ	<i>measure, Lebesgue measure</i>
$P(A)$	<i>probability of the event A</i>
$\mathbb{E}(X), \operatorname{Var}(X)$	<i>expectation, variance of a random variable</i>
$\mathbf{1}_A$	<i>indicator function of A</i>
$ G $	<i>order of a finite group G</i>
G/H	<i>quotient group, for H normal in G</i>
$\mathbb{Z}/n\mathbb{Z}$	<i>integers modulo n</i>
$a \equiv b \pmod{n}$	<i>congruence modulo n</i>
$\cong$	<i>isomorphic to</i>
$\chi_A(\lambda)$	<i>characteristic polynomial of A</i>
$\sigma(T)$	<i>spectrum of the operator T</i>
$\hat{f}, \mathcal{F}(f)$	<i>Fourier transform of f</i>
$\operatorname{Res}(f, a)$	<i>residue of f at a</i>
$\mathcal{A}, \sigma(\mathcal{C})$	<i>σ-algebra, σ-algebra generated by $\mathcal{C}$</i>

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FROM ELEMENTS TO FOUNDATIONS

There comes a moment, in the study of mathematics, when answers cease to be the goal and questions take their place. That moment is the one you are crossing now.

In secondary school, a theorem was a tool: one applied it to solve an exercise, then moved on to the next. In undergraduate mathematics, a theorem becomes an object: one proves it, seeks its limits, asks what happens when its hypotheses are weakened. The derivative, which you know how to compute, will be refounded on the completeness of the reals. The vectors you manipulated in the plane will live in spaces of arbitrary dimension. Randomness, which you could quantify, will rest on the axioms of measure.

It is not that mathematics becomes harder. It becomes deeper. And depth, once reached, simplifies everything: it is by understanding why a result is true that one ceases to need to memorize it.

The grammar of reality, whose first words were laid down in the Elements, now awaits its syntax.

Part I

Learning to Read Mathematics

The volume begins with language. Statements and their negation, the connectives of implication and equivalence, and the universal and existential quantifiers fix what a mathematical sentence actually asserts; sets, with membership and inclusion, supply the objects; maps relate them, with injectivity, surjectivity, and bijectivity. These conventions carry every definition and every proof in the volume.

CHAPTER I

WHAT A MATHEMATICAL STATEMENT IS

Before computing anything at all, one must learn to say something that can be judged true or false. This chapter lays the first stone: what is a mathematical statement, and how does it differ from an ordinary sentence? Everyday language tolerates vagueness, implicit meaning, productive ambiguity; mathematics demands that every word be held to account. By learning to formulate, and then to negate, a statement, the reader discovers that rigor is not a constraint: it is the very condition under which thought can advance without stumbling.

Articles in this chapter:

- 001.** *When Ordinary Language Falls Short — The flexibility of everyday language becomes dangerous when precise reasoning is required.*
- 002.** *Saying Something That Can Be Judged — Symbols come later; before them, there are statements: sentences one can judge.*
- 003.** *What a Sentence Often Lacks — A sentence may be neither absurd nor a statement: it is simply incomplete.*
- 004.** *A Mathematical Truth Is Not an Impression — “one can see that” has no mathematical value: one must be able to prove.*
- 005.** *False Statement or Ill-Defined Expression — Not all difficulties are on the same level: false $\neq$ ill-defined.*
- 006.** *Mathematics Begins with Distinctions of Language — No major theorem has been proved in this chapter.*
- 007.** *First Operation on a Statement: Negation — One negates neither a command nor an ill-defined expression: one negates a determinate assertion.*

I. WHEN ORDINARY LANGUAGE FALLS SHORT.

ONE word can change which numbers we keep. Suppose we are asked to select the integers “divisible by 2 or by 3”. Should we include 6? It is divisible by both. The answer depends on what the word “or” allows.

If the instruction requires *at least one of the two properties*, 6 qualifies: satisfying both does not disqualify it. This is inclusive “or”, the meaning used in mathematics unless stated otherwise.

If the instruction requires *exactly one of the two properties*, 6 no longer qualifies. We must then write: “divisible by 2 or by 3, but not by both”. This is exclusive “or”.

The numbers 2 and 3 qualify under either reading; 5 qualifies under neither. It is 6 that reveals the difference. No difficult calculation was needed: we had to know which condition was being imposed.

In conversation, context often supplies what is left unsaid. In mathematics, it can fix the domain, the objects, and the conventions; these must be recoverable without ambiguity. A sentence is read with the assumptions governing it.

REMARK 1.1: Mathematics does not demand nobler words or longer sentences. It demands words used *consistently*: each term must retain the meaning fixed in its context. It does not require that everything be complicated. It requires that the *distinctions* be maintained: true or false, well defined or ill defined, proved or unproved. This does not mean that we can decide the truth of every statement.

The first mathematical act is therefore not to calculate. It is to formulate something that can be judged, and that is already an art. In the following exercises, no variable value has already been fixed unless stated otherwise. Having a truth value does not mean that we know how to determine it.

The first exercise asks the reader to identify ambiguity in everyday language. The second asks whether a mathematical expression is precise enough to have a truth value.

- *Among the integers from 1 to 12, list those divisible by 2 or by 3: (1) by at least one of the two numbers; (2) by exactly one. What is the difference between these two readings?*

Solution: *Inclusive or*: The list is 2, 3, 4, 6, 8, 9, 10, 12. *Exclusive or*: The list is 2, 3, 4, 8, 9, 10. The integers 6 and 12, divisible by both 2 and 3, have been removed. The integers 1, 5, 7, 11 meet neither condition. In mathematics, "or" ordinarily includes the case where both statements hold. To exclude it, say "or, but not both."

- *"The number π is about 3.14." Is this a mathematical statement? Why?*

Solution: No. A mathematical statement must be true or false without vagueness. "About" does not allow a verdict as long as no convention fixes the intended precision; the sentence is insufficiently defined, neither true nor false. One could instead write " $3.14 < \pi < 3.15$ " (true) or " $\pi = 3.14$ " (false). Those are statements; "about" is not one. Mathematics requires precise terms, not verbal approximations.

2. SAYING SOMETHING THAT CAN BE JUDGED.

BEFORE the symbols, before the formulas, there is a more elementary act: uttering a sentence of which one can say whether it is true or false. Many students believe that mathematics consists mainly in manipulating symbols. Symbols come later. Before them, there are *statements*: sentences that can be judged.

Consider five sentences:

$$2 + 3 = 5 \quad ; \quad 7 \text{ is prime} \quad ; \quad x + 1 > 0.$$

"Close the door." "What will the weather be tomorrow?"

The first two are mathematical statements in the simplest sense: each asserts something, and that assertion can be judged true or false. The third looks like a statement, but it remains suspended as long as one does not know what x is. The fourth is a command. The fifth is a question. Neither lends itself to the verdict true or false.

DEFINITION 2.1: In the classical logic used here, a *proposition* or *mathematical statement* is an affirmative sentence which,

once its context and interpretation are fixed, has a truth value: true or false. That value may be unknown to us.

Mathematical reasoning does not operate on arbitrary sentences. It operates on statements. From them one builds proofs, draws consequences, and identifies errors. Statements are, so to speak, the raw material of reasoning.

REMARK 2.2: Other texts sometimes use the word *assertion*. In this volume, we shall mainly use “*statement*”, but all three terms designate the same logical object.

The definition is short. Yet it imposes a strict discipline: a vague sentence is not yet a statement, a suggestive sentence is not yet a statement, an incomplete sentence is not yet a statement. Recognizing what is a statement and what is not is the first step toward reasoning, and perhaps the most decisive one. In the classical logic of this volume, bivalence thus provides an initial foothold for reasoning; it is distinct from our ability to find a proof or a refutation.

The first exercise classifies sentences as statements or non-statements. The second asks for the truth value of a statement.

- *Classify each sentence: statement, question, command, or incomplete sentence. (1) $3 + 4 = 8$. (2) Solve $x^2 = 4$. (3) Every square is nonnegative. (4) n is even. (5) What is 7×8 ?*

Solution: (1) Statement, false: $3 + 4 = 7$. (2) Command: no true/false verdict. (3) Statement, true: $x^2 \geq 0$ for every real x . (4) Incomplete sentence: we do not know which n is being discussed. One may fix n , for example $n = 2$, or quantify, for instance, “for every integer n ” (false) or “there exists an integer n such that n is even” (true). (5) Question: no verdict. Only (1) and (3) are statements.

- For each statement, give its truth value. (1) $\sqrt{4} = 2$. (2) $0.999 \dots = 1$. (3) There exists an integer n such that $n^2 = 2$.

Solution: (1) True: by convention, $\sqrt{4}$ denotes the nonnegative square root, namely 2. (2) True: 0.999... = 1; it is not almost 1, it is exactly 1; the finite sum $\sum_{n=1}^{\infty} 10^{-n} = 1/9$ gets as close to 1 as desired, and the number 0.999... is by definition the limit of these sums, which equals 1 (infinite series are constructed later). (3) False: $\sqrt{2}$ is irrational, so no integer n satisfies $n^2 = 2$. The statement is well formed and false; it is neither absurd nor ill-defined.

3. WHAT A SENTENCE OFTEN LACKS.

HERE is an expression everyone recognizes:

$$x^2 - 1 = 0.$$

It is not absurd. But taken on its own, with no value of x fixed by the context, it is not yet a statement. Does one claim it holds for every real number x ? That there exists at least one x satisfying it? That one seeks the values of x for which it holds? Without this precision, the sentence hangs in the air, like a question missing its question mark.

By contrast, the following two sentences are genuine statements:

“There exists a real number x such that $x^2 - 1 = 0$.”

“For every real number x , we have $x^2 - 1 = 0$.”

The first is true: take $x = 1$. The second is false: take $x = 0$ to produce a counterexample. The expression $x^2 - 1 = 0$ is the same in both cases. What changes are the words that frame it, and those words reverse the verdict.

REMARK 3.1: In mathematics, the most inconspicuous terms are often the most decisive. “There exists” and “for all” are not commentary appended to an expression: they fix its scope. The first localizes, the second universalizes. If x has not already been fixed, a quantifier and a domain give $x^2 - 1 = 0$ its missing scope. Fixing $x = 1$ is another option: the equality is then true. In the two preceding statements, changing the quantifier reverses the truth value; this does not always happen. This opposition between the local and the global runs through all of mathematics: asserting the existence of an object satisfying a condition, or asserting that the

condition holds everywhere, are two fundamentally different acts, and it is their distinction that makes reasoning possible.

We shall return at length to these two constructions, which are called *quantifiers*. For now, it suffices to retain this: a sentence may have every appearance of a mathematical statement and yet fail to be one. What it lacks is the frame that fixes its meaning: a value already fixed, or quantification over a specified domain. Without this frame, the sentence drifts between true and false without settling anywhere, and reasoning cannot even begin.

The first exercise completes an incomplete sentence in two opposite ways. The second identifies what a sentence lacks in order to become a statement.

- The sentence “ $x^2 > x$ ” is incomplete. Complete it in two ways, one giving a true statement and one giving a false statement.

Solution: True: “there exists a real number x such that $x^2 > x$ ”; take $x = 2$, since $4 > 2$. False: “for every real number x , $x^2 > x$ ”; take $x = 1/2$, since $1/4 < 1/2$. The same expression gives a true or false statement depending on the quantifier. The frame, for all or there exists, fixes the meaning, not the expression by itself.

- What is missing from each sentence for it to become a statement? (1) $2n+1$ is odd. (2) f is increasing. (3) $A \cap B = \emptyset$.

Solution: (1) Without prior context, one must fix n or quantify it over a domain: “for every integer n ” (true), or “there exists an integer n such that ...” (also true, but weaker). (2) The function f is missing which function? The interval is also missing on which domain? (3) The definitions of A and B are missing, which sets? In all three cases the sentence is grammatically correct but logically suspended; it says nothing that can yet be judged.

4. A MATHEMATICAL TRUTH IS NOT AN IMPRESSION.

ONE hears it often: “it is obvious”, “one can clearly see that”, “it must be true”. These phrases sometimes serve a psychological purpose. They carry no mathematical weight. In mathematics, conviction is not a substitute for proof. Justifying a statement is not finding it plausible. One may rely on an established result or derive it from definitions, hypotheses, and accepted facts. Truth in an interpretation and provability in a fixed system remain distinct notions.

Three levels must be kept apart, though they are routinely conflated: a sentence that is *plausible*, a sentence that is *true*, and a sentence that is *proved*.

EXAMPLE: All odd numbers are prime.

This statement is perfectly well-formed. One can assign it a truth value. It is false. Consider $9 = 3 \times 3$: an odd number that is not prime. The verdict here is swift. But reaching it required exhibiting a counterexample, not trusting an impression.

EXAMPLE: The sum of two even numbers is even.

This statement is true. Yet to enter a rigorous argument, one must still know *why* it is true. An elementary proof suffices: if $a = 2m$ and $b = 2n$, with m, n integers, then

$$a + b = 2m + 2n = 2(m + n),$$

so that $a + b$ is even. The passage from intuition to proof is like the passage from a rough sketch to an architectural drawing: the aim is not to complicate, but to make visible the reason the statement holds.

REMARK 4.1: A mathematical truth is not enough on its own. To serve as a foothold for reasoning, it must take its place in a network of reasons: definitions, hypotheses, logical steps. This network is what separates a conviction from a proof, and it is where the entire operative force of mathematics resides.

Learning to prove is therefore learning to account for one's reasons, and this demand, far from constraining thought, is what gives it reach.

The first exercise distinguishes impression from proof. The second asks for a counterexample refuting a plausible statement.

- “The sum of two odd numbers is odd.” True or false? Justify.

Solution: False. $3 + 5 = 8$, which is even. More generally, if $a = 2p + 1$ and $b = 2q + 1$, with p, q integers, then $a + b = 2(p + q + 1)$, which is even. The impression “odd plus odd is odd” is natural but false. A two-line proof settles the question. This is the difference between an impression and a mathematical truth.

- “If n is prime, then $2^n - 1$ is prime.” Test $n = 2, 3, 5, 7$. Then try $n = 11$.

Solution: $n = 2$: 3, prime. $n = 3$: 7, prime. $n = 5$: 31, prime. $n = 7$: 127, prime. Four successes in a row. But for $n = 11$, $2^{11} - 1 = 2047 = 23 \times 89$, not prime. The impression that it works for primes was false. A single counterexample is enough to refute a universal statement. Numbers $2^n - 1$ that are prime are called Mersenne primes; it remains unknown whether infinitely many exist.

5. FALSE STATEMENT OR ILL-DEFINED EXPRESSION.

A sentence that fails can fail in two very different ways. Confusing these two failures means understanding neither. A statement can be false: it asserts something precise, but that assertion is incorrect. An expression can also fail upstream, before it even produces a statement: it purports to designate an object, but that designation collapses.

EXAMPLE: The largest prime number exists.

This sentence is well-formed. It asserts something clear: there exists a prime number greater than every other. It is a statement. It happens to be false, since Euclid showed that there are infinitely many primes. But at least the sentence says something one can judge.

COUNTEREXAMPLE: The real number closest to zero without being equal to zero.

Here the situation is altogether different. This group of words says nothing true or false. It attempts to designate an object, but the designation fails: there is no nonzero real number whose absolute value is strictly smaller than that of every other nonzero real number: however close to zero one chooses a real $r \neq 0$, the real $r/2$ (of the same sign as r) is closer still. The expression therefore fails to pin down an object. It is a verbal mirage: the phrase appears to point at something, but there is nothing at the end of the finger.

REMARK 5.1: One must learn to recognize these two situations. In the first case, the sentence is a statement that asserts something incorrect. In the second, the expression does not even manage to pose correctly what it claims to designate. A sharp sentence can be false. A reasonable-looking expression can be ill-defined. Knowing how to distinguish these two failures is one of the first competencies that mathematical reasoning demands, and one of the least taught.

Before judging whether a sentence is true or false, one must first verify that it manages to say something at all, and it is this vigilance that opens the way to rigorous definitions.

The first exercise distinguishes false from ill-defined. The second identifies an expression that denotes nothing.

- *Classify: false or ill-defined? (1) $\sqrt{2}$ is rational. (2) The smallest strictly positive real number. (3) $1 = 0$.*

Solution: (1) False statement: $\sqrt{2}$ is well defined as a real number, and the assertion that it is rational is false. (2) Ill-defined: the expression tries to name an object that does not exist, for every $r > 0$, the number $r/2$ is still smaller and still positive. It is neither true nor false: it is a mirage. (3) False statement: 1 and 0 are well defined, and the assertion $1 = 0$ is simply false. The distinction is crucial: (1) and (3) say something false; (2) fails to say anything determinate.

- *“The largest prime number.” Is this a well-defined object? Justify.*

Solution: No. There is no largest prime number, by Euclid's theorem: given any finite list of primes, one can find a prime divisor that is not on the list. The expression looks like a name, as if it pointed to a precise object, but it denotes nothing. It is an ill-defined expression, not a false statement. The statement “there exists a largest prime number” is well formed and false.

6. MATHEMATICS BEGINS WITH DISTINCTIONS OF LANGUAGE.

No major theorem has been proved in the articles so far. No formula has been established for its own sake: the few computations we met served only to show what a proof is. Yet something essential has been laid down: the distinctions without which no theorem can be read correctly.

A proposition is not a question. A statement is not a command. A complete sentence is not an incomplete one. A proved truth is not

a plausible impression. A well-defined object is not a reasonable-looking expression that fails to designate anything.

REMARK 6.1: These distinctions seem simple. They are simple in their formulation. They are far less simple in practice, because they foreshadow a deeper principle: in mathematics, the first step of every construction is to fix without ambiguity what one is talking about. Defining a set, stating a theorem, conducting a proof: everything begins with a sentence whose logical status is clear. This is exactly the work that axiomatics will take up on a larger scale: laying down rules of language so precise that reasoning can proceed from an explicit context, without depending on tacit intuition.

Mathematics is not exacting out of a taste for complication. It is exacting because it refuses the silent slippages that ordinary language tolerates. The work of these first articles has been to learn to notice those slippages; the next article takes up the first operation on statements, negation. It is modest work in appearance, but everything that follows rests upon it, as a building rests on its foundations, invisible yet load-bearing.

This article is a synthesis. The exercise recaps the distinctions of the chapter by classifying a list of sentences.

- *Classify each sentence in the right category: true statement, false statement, incomplete sentence, command, question, ill-defined expression. (1) $2^{10} = 1024$. (2) Compute $3!$. (3) $x > 5$. (4) The real number whose square is -1 . (5) Every even number is divisible by 4. (6) What is the value of π ?*

Solution: (1) True statement: $2^{10} = 1024$. (2) Command. (3) Incomplete sentence: which x ? (4) Ill-defined expression. (5) False statement: 0 is even but not divisible by 4. (6) Question. These six categories cover the statuses distinguished in this exercise.

7. FIRST OPERATION ON A STATEMENT: NEGATION.

ASSERTING is a first act. The second, equally fundamental, is contradicting. As soon as a statement is well-formed, one can apply a simple logical operation to it: negate it. But one must be precise: one negates neither a command nor an ill-defined expression. One negates a determinate assertion, and one obtains a new determinate assertion.

PROPOSITION 7.1: The negation of a mathematical statement is again a mathematical statement.

COMMENTED PROOF: Let P be a statement. Only two cases arise: P is true or P is false. If P is true, then “not P ” is false. If P is false, then “not P ” is true. In both cases, the sentence “not P ” receives a truth value. It is therefore a statement.

The principle is clear. Putting it into practice, however, holds surprises.

EXAMPLE: *2 is even. Negation: 2 is not even.*

Here the negation is immediate: one replaces the property by its contrary.

EXAMPLE: *There exists an integer n such that $n^2 = 9$. Negation: for every integer n , $n^2 \neq 9$.*

To negate the existence of an object satisfying a condition, one asserts that *all* objects fail to satisfy it.

EXAMPLE: *For every real number x , $x^2 \geq 0$. Negation: there exists a real number x such that $x^2 < 0$.*

The negation of a “for all” becomes a “there exists”. One does not negate this sentence by writing “for every real number x , $x^2 < 0$ ”: that would assert that *all* squares are negative, which is far stronger than the correct negation.

This last example reveals an essential point: one does not negate a statement by slipping in a “not”. One must look at the logical structure of the sentence. When a “for all” appears, negation

transforms it into a “*there exists*”, and conversely. Negation is not a mechanical operation on words; it is an operation on structure. For statements P, Q , with inclusive “or”, the same reading gives the following rules (each equivalence means that both sides always have the same truth value):

$$\begin{aligned}\neg\neg P &\equiv P, \\ \neg(P \wedge Q) &\equiv (\neg P) \vee (\neg Q), \\ \neg(P \vee Q) &\equiv (\neg P) \wedge (\neg Q).\end{aligned}$$

The first follows by negating twice. For the other two, called De Morgan’s laws, denying that both statements are true means at least one is false; denying that at least one is true means both are false. Their set-theoretic translation returns in chapter 2.

REMARK 7.2: The interchange between “*for all*” and “*there exists*” under negation is a first instance of duality: two notions that appear distinct turn out to be linked by an operation transforming one into the other. This symmetry will reappear in De Morgan’s laws (for sets, the complement of a union is the intersection of the complements, and conversely) and, much further on, in the duality between a space and its dual. Recognizing such symmetries is one of the guiding threads of all mathematics.

COUNTEREXAMPLE: “*Find a prime number greater than 10.*”

This sentence is not a statement: it is a command. One cannot negate it.

Negation is the first tool that logic offers for transforming a statement, and learning to use it correctly is already learning to reason.

The first exercise writes the negation of simple statements. The second negates quantified statements by switching for all and there exists.

- Write the negation of: (1) 5 is even. (2) $\sqrt{9} = 3$. (3) Every triangle has three sides.

Solution: (1) 5 is not even, or equivalently 5 is odd. True. (2) $\sqrt{9} \neq 3$. False, by definition of a triangle. In each case the negation is a new statement exists a triangle that does not have three sides. False. (3) For every integer n , $2^n > n$. True. (4) There exists a triangle that does not have three sides. False.

- Write the negation of: (1) For every real x , $x^2 + 1 > 0$. (2) There exists an integer n such that $n^2 = 5$. (3) For every integer $n \geq 1$, $2^n > n$.

Solution: (1) There exists a real x such that $x^2 + 1 \leq 0$. False, since always $x^2 + 1 \geq 1 > 0$. (2) For every integer n , $2^n \neq 5$. True, because $\sqrt{5}$ is not an integer. (3) There exists an integer $n \geq 1$ such that $2^n \leq n$. False; induction gives $2 < 1$, then $2^n > n + 1$ for $n \geq 1$. Rule: "for every" becomes "there exists" and the property is negated. "There exists" becomes "for every" and the property is negated.

METHODS

Study 1. ★ METHOD: recognize a mathematical statement

A statement is a sentence to which one can assign true or false. It is not a question, not a command, not an incomplete sentence with a variable neither fixed by context nor quantified, and not an ill-defined expression. When reading a sentence, first ask: can it be judged true or false? If yes, it is a statement. If not, identify what is missing or what fails.

Among the following sentences, which are statements? For each non-statement, explain why. (i) $7 \times 8 = 54$. (ii) Let x be a real number. (iii) $n! > 2^n$. (iv) The circle of radius -1 .

- Classify each sentence.
- For the statements, give the truth value.
- Summarize the method.

Solution: (i) Statement: it is a judicable assertion. (ii) Introduction of a variable: neither true nor false; it is a declaration, not a statement. (iii) Incomplete sentence: which n ? It is not a statement until n is fixed or quantified. (iv) Ill-defined expression: a radius is a nonnegative length, so -1 is not a valid radius. (b) (i) False: $7 \times 8 = 56$. (c) When facing a sentence: first ask whether true or false can be assigned. If yes, it is a statement. If not, ask whether it is a question, a command, an incomplete sentence, or an ill-defined expression.

Study 2. ★ METHOD: *distinguish false from ill-defined*

A false statement says something precise but inaccurate. An ill-defined expression fails to designate an object. Before judging true or false, check that every term in the sentence really designates an object. If a term fails to designate, the sentence is ill-defined, not false.

Classify: false or ill-defined? (i) π is rational. (ii) The least element of $(0, 1)$. (iii) $(-1)! = -1$. (iv) The set of all sets, in the usual Zermelo–Fraenkel framework.

- (a) *For each sentence, identify whether the terms are well defined.*
- (b) *Classify.*
- (c) *Summarize.*

Solution 2: (a) (i) π is well defined and “rational” is well defined; the terms hold. (ii) (iii) (iv) $(0, 1)$ is well defined, but “the least element” fails, because for every $r \in (0, 1)$, $r/2$ is still in $(0, 1)$. (iii) $(-1)!$ is not defined in the usual elementary convention, where factorial is defined for integers ≥ 0 . (iv) Russell’s paradox shows that this collection cannot be a set; the unlimited comprehension of naive set theory leads to a contradiction, and axiomatic set theory restricts it precisely to avoid this (the explication of the barber paradox, at the end of this chapter, gives the argument). (b) (i) False. (ii) Ill-defined. (iii) Ill-defined. (iv) Ill-defined. (c) Before true/false, verify that every term designates something. If one term fails, the phrase is ill-defined.

Study 3. ★★ METHOD: *write the negation of a statement*

To negate a statement, keeping the variable domains unchanged: (1) identify the logical structure, including quantifiers and connectives; (2) switch the quantifiers, $\forall \leftrightarrow \exists$; (3) negate the inner property, so = becomes $\neq$, $>$ becomes $\leq$, and so on. Do not merely insert “not” into the sentence: that can produce a statement stronger than the correct negation.

Write the negation of: (i) For every integer $n \geq 1$, $n! \geq 2^{n-1}$. (ii) There exists a real x such that $\sin x = 2$. (iii) Let (u_n) be a real sequence and ℓ a real number: for every $\varepsilon > 0$, there exists a natural number N such that for every natural number $n \geq N$, $|u_n - \ell| < \varepsilon$.

- (a) *Identify the structure of each statement.*
- (b) *Apply the rules of negation.*
- (c) *Summarize.*

Solution 3: (a) (i) $\forall n \geq 1, p(n)$. (ii) $\exists x, p(x)$. (iii) $\exists \varepsilon > 0, \forall N, \forall n \geq N, p(n, \varepsilon)$; this is the definition of convergence. (b) (i) There exists $n \geq 1$ such that $n! < 2^{n-1}$. False; for $n = 1$, both sides equal 1; for $n > 2$, each of the $n - 1$ factors in $2 \cdot \dots \cdot n$ is at least 2. (ii) True, since $|\sin x| \leq 1$. (iii) There exists $\varepsilon > 0$ such that for every natural number N , there exists a natural number $n \geq N$ such that $|u_n - \ell| \geq \varepsilon$. This is the definition of “ u_n does not converge to ℓ ”. (c) Rule: switch each quantifier from left to right, then negate the final property.

Study 4. ★★ METHOD: *refute a statement by counterexample*

To show that a universal statement, $\forall x, P(x)$, is false, a single counterexample is enough: exhibit an x_0 for which $P(x_0)$ is false. A counterexample does not prove that P is always false; it proves that P is not always true. That is the exact negation of $\forall x, P(x)$.

Refute each statement by a counterexample. (i) For every integer $n \geq 2$, $2^n - 1$ is prime. (ii) For all real x, y , $|x + y| = |x| + |y|$. (iii) Every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable.

- (a) Find a counterexample for each statement.
 (b) Check that the counterexample satisfies the hypotheses but not the conclusion.
 (c) Summarize.

Solution 4: (a) (i) $n = 4$; $2^4 - 1 = 15 = 3 \times 5$, not prime. (ii) $x = 1, y = -1$; $|1 + (-1)| = 0$, but $|1| + |-1| = 2$, so $0 \neq 2$. (iii) $f(x) = |x|$: continuous at 0 but not differentiable at 0, because the difference quotient has right limit $+1$ and left limit -1 . (b) (i) $n = 2$: hypothesis true, conclusion false. (ii) $x = 1, y = -1$: real numbers, conclusion false. (iii) $|x|$ is continuous everywhere, but not differentiable at 0. (c) One counterexample destroys a universal statement. But 1000 favorable examples do not prove it unless they exhaust the domain, on a finite domain, exhaustive checking can constitute a proof.

Study 5. ★★ METHOD: *complete a sentence into a statement*

A sentence containing a variable such as x , n , or f , neither fixed by context nor quantified, remains open. To complete it, one may fix its value or: (1) add a quantifier, $\forall$ or $\exists$; (2) specify the domain of the variable, such as integer, real number, function, and so on. The choice of quantifier changes the meaning: $\forall$ universalizes, whereas $\exists$ localizes. The same predicate can give a true or false statement depending on the quantifier.

Complete " $n^2 - 4n + 3 = 0$ " in four ways: a true statement with $\exists$, a false statement with $\exists$, a true statement with $\forall$, and a false statement with $\forall$.

- (a) Find the roots of $n^2 - 4n + 3 = 0$.
 (b) Build the four statements.
 (c) Summarize.

Solution 5: (a) $n^2 - 4n + 3 = (n - 1)(n - 3) = 0$, so the roots are $n = 1$ and $n = 3$. (b) True with $\exists$: "there exists an integer n such that $n^2 - 4n + 3 = 0$ ", for instance $n = 1$. False with $\forall$: "for every integer n such that $n^2 - 4n + 3 = 0$ ", since both roots are ≤ 10 . True with $\forall$: "not natural over a large domain, but true over the restricted domain $\{1, 3\}$." (c) "for every $n \in \{1, 3\}$, $n^2 - 4n + 3 = 0$ ". False with $\forall$: "for every integer n , $n^2 - 4n + 3 = 0$ ", since $n = 0$ gives $3 \neq 0$. (c) The quantifier and the domain fix the meaning; the same expression can become true or false.

Study 6. ★★ METHOD: *analyze the logical structure of a definition*

A mathematical definition fixes the meaning of a term unambiguously. To check whether a definition is correct: (1) is every word held in a single sense? (2) Is every variable quantified or fixed? (3) If the definition purports to name a specific object ("the X such that..."), does that object exist? (4) Is it unique? (A definition of a predicate or a set stays correct even without an instance: the empty set is a well-defined set.) A definition is not true or false: it is well formed or badly formed.

Analyze the definition: "A prime number is an integer greater than or equal to 2 whose only positive divisors are 1 and itself."

- (a) *Is the definition well formed? Is every term held?*
- (b) *Check it on examples: 7, 9, 1, 2.*
- (c) *Why is 1 excluded? What would happen if it were included?*

Solution 6: (a) Well formed. "integer" has a fixed meaning, "greater than or equal to 2" is clear, and "positive divisors are 1 and itself" is decidable. (b) 7: divisors 1, 7, so prime. 9: divisors 1, 3, 9, so not prime. 1: divisors {1} only, but excluded by the condition ≥ 2 . 2: divisors 1, 2, so prime; it is the only even prime. (c) If 1 were included, the fundamental theorem of arithmetic, uniqueness of prime factorization, would fail: $6 = 2 \times 3 = 1 \times 2 \times 3 = 1^2 \times 2 \times 3$, and so on. Excluding 1 is not arbitrary; it is needed for unique factorization.

METHOD RECAP

RECOGNIZING A STATEMENT: *Can one say true or false? If yes, it is a statement. Otherwise: question, command, declaration (introduction of a variable), incomplete sentence, or ill-defined expression.*

FALSE VS ILL-DEFINED: *False = says something inaccurate. Ill-defined = fails to designate an object. Check the terms before judging.*

NEGATION: $\forall \leftrightarrow \exists$, then negate the property. Do not insert "not" mechanically.

COUNTEREXAMPLE: *One counterexample is enough to refute $\forall x, P(x)$. A thousand examples do not prove $\forall$ unless they cover the whole domain.*

COMPLETING A STATEMENT: *Unfixed variable $\rightarrow$ fix its value or add quantifier plus domain. The choice may change the truth value.*

DEFINITION: *Every word held, every variable fixed; if a specific object is named, it must exist (and be unique when required).*

EXPEDITIONS

EXPEDITION I: THE BARBER PARADOX

Prerequisites: studies 2, 6.

In a village, a barber who is himself a resident shaves exactly those residents who do not shave themselves. Does the barber shave himself?

- (a) *Suppose the barber shaves himself. What does the rule say?*
- (b) *Suppose he does not shave himself. What does the rule say?*
- (c) *The two assumptions lead to a contradiction. What is the conclusion?*

Hint: If the barber shaves himself, he is among those who shave themselves, so the barber should not shave him. Contradiction. If the barber does not shave himself, he is among those whom the barber must shave, so he should shave himself. Contradiction.

- (d) *Compare this with the expression “the set of all sets that do not contain themselves”. This is Russell’s paradox (1901).*
- (e) *What does this paradox teach us about definitions?*

The barber paradox is the popular version of Russell’s paradox: the set $R = \{x : x \notin x\}$ cannot exist without contradiction, since $R \in R \Leftrightarrow R \notin R$. This paradox triggered a crisis in foundations in the early 20th century and led to axiomatic set theory, such as Zermelo–Fraenkel, which restricts set formation: one no longer freely forms $\{x : P(x)\}$, the separation schema selects the elements satisfying P inside an already given set. Other axioms allow other constructions. The lesson for this chapter: a reasonable-looking expression can be ill-defined. Everyday language lets us formulate sentences that seem meaningful but contain destructive self-reference. Mathematics

rules out this uncontrolled self-reference, the kind that produces a contradiction with no rule to contain it, and this discipline is what avoids this precise type of contradiction. (Carefully controlled self-reference, by contrast, can be fruitful: Gödel’s theorem, later in this chapter, gives a central example.)

Report 1: (a) If he shaves himself, he is among those who shave themselves, but the barber shaves only those who do not shave themselves: contradiction. (b) If he does not shave himself, he is among those who shave themselves, but the barber shaves only those who do not shave himself: contradiction. (c) Two readings must be distinguished. The statement “such a barber exists” is simply *false*: no individual can satisfy the rule, which would require him both to shave himself and not to shave himself (the predicate itself is self-referential). This is not a paradox of the real world but of language: the sentence seems well formed but contains circular self-reference. (d) If a universal set U existed in this framework, separation would give $R = \{x \in U : x \notin x\}$. Since $R \in U$, we would have $R \in R \Leftrightarrow R \notin R$ and $R \notin R \Leftrightarrow R \in R$, the same structure. (e) Every definition must be checked: does it designate an object that exists without contradiction? The paradox shows that a grammatically correct sentence can be logically impossible.

EXPEDITION 2: FORTY EXAMPLES ARE NOT A PROOF

Prerequisites: studies 4.

The polynomial $P(n) = n^2 + n + 41$ gives prime numbers for $n = 0, 1, 2, \dots, 39$. Is that a proof that $P(n)$ is always prime?

- (a) Check that $P(0) = 41$, $P(1) = 43$, $P(2) = 47$, and $P(3) = 53$ are all prime.
- (b) Compute $P(40)$. Factor it.

Hint: $P(40) = 1600 + 40 + 41 = 1681 = 41^2$.

- (c) Why did $P(40)$ have to be composite? Compute $P(41)$.

Hint: $P(41) = 41^2 + 41 + 41 = 41^2 + 2 \times 41 = 41(41 + 2) = 41 \times 43$.

- (d) What does this example prove about the relation between experimental checking and proof?
- (e) Is there a polynomial that produces only primes for every natural number n ?

Euler’s polynomial $n^2 + n + 41$ is prime for $n = 0, \dots, 39$, forty consecutive values, but $P(40) = 41^2$ is composite. This is one of the clearest examples of the distinction between verification and proof: forty confirmations are not a proof, and a single failure refutes. In fact, no nonconstant polynomial with integer coefficients can produce only prime numbers as n ranges over $\mathbb{N}$. If $P(a) = p$ is prime, then the values $P(a + kp)$ are divisible by p for every integer $k \geq 0$; since P is nonconstant, $|P(a + kp)|$

eventually exceeds p as $k \rightarrow +\infty$, so $P(a + kp)$ cannot be prime beyond some point, being divisible by p with absolute value larger than p . The lesson is deep: empirical induction by observation is not a proof. It must not be confused with mathematical induction, which is a deductive proof method. Deduction, chaining implications from hypotheses, is proof.

Report 2: (a) $P(0) = 41$, $P(1) = 43$, $P(2) = 47$, and $P(3) = 53$, all prime. The run continues: all prime up to $P(39) = 1601$. (b) $P(40) = 1600 + 40 + 41 = 1681 = 41^2$, composite. (c) $P(40) = 40^2 + 40 + 41 = 40(40 + 1) + 41 = 40 \times 41 + 41 = 41(40 + 1) = 41^2$. Also $P(41) = 41^2 + 2 \times 41 = 41(41 + 2) = 41 \times 43$. The 41 in the polynomial guarantees divisibility. (d) Forry favorable examples do not prove the universal claim. One counterexample refutes it. Empirical induction is not mathematical proof. (e) Not in any interesting way: a constant polynomial $P(n) = p$ with p prime trivially works, but no nonconstant polynomial with integer coefficients produces only primes as n ranges over $\mathbb{N}$. The elementary proof is that if $P(a) = p$ is prime, then $P(a + p), P(a + 2p), \dots$ are all divisible by p , and, since P is nonconstant, eventually exceed p in absolute value; divisible by p with absolute value larger than p , they are no longer prime.

EXPEDITION 3: THE SENTENCE THAT SPEAKS ABOUT ITSELF

Prerequisites: studies 1, 2.

“This sentence is false.” True or false?

- (a) *If the sentence is true, what does it say about itself?*
- (b) *If the sentence is false, what does it say about itself?*
- (c) *Why can we not assign it true or false? Which category from the chapter does it violate?*

Hint: The sentence is not a mathematical statement: it does not satisfy the definition, because one cannot assign it true or false without contradiction.

- (d) *Compare it with “ $2 + 2 = 5$ ”, false but not paradoxical, and with “ $x > 3$ ”, incomplete but not paradoxical. Why is the liar paradox different?*
- (e) *Gödel (1931) constructed an arithmetical statement that says of itself that it is unprovable in a given formal system. Is that statement a paradox?*

“This sentence is false” is the liar paradox, known since antiquity (Eubulides of Miletus, 4th century BC). In a classical reading that identifies the truth of the sentence with what it asserts, no consistent assignment of true or false is possible: assigning either value forces the other. It is therefore not a statement in the sense of article 2. The difficulty here is self-reference applied to its own falsehood, not self-reference in general. In 1931, Gödel transformed this idea into a rigorous theorem, which we admit here (its proof belongs to mathematical logic): in every consistent formal system whose proofs can

be checked mechanically and which is rich enough to contain arithmetic, there are statements true in the standard model of the integers but not provable in the system. This sharpens the distinction in article 4: truth in an interpretation is distinct from provability in a fixed system. The truth of the Gödel sentence is established here by reasoning outside the system being studied, assuming its consistency; this is not a proof produced by that system. Gödel's statement is not a paradox; it is true, not contradictory. It shows that mathematical truth can exceed what any given formal system can prove. Chapter 1 of this volume asked what a statement is. Gödel shows that the question is deeper than it first appears.

Report 3: (a) If "this sentence is false" is true, then what it says is true, then what it says is false. Contradiction. (b) If it is false, then what it says is incorrect: it is not false, so it is true. Contradiction. (c) In this classical reading, neither true nor false can be assigned consistently; it violates the definition of a statement in article 2. It is not a mathematical statement. The source is uncontrolled self-reference, which directly produces a contradiction. (Mathematically controlled self-reference, without that contradiction, also exists: that is exactly the point of Gödel's theorem above.) (d) $2 + 2 = 5$ is false but nonparadoxical, because there is no self-reference. "x > 3" is incomplete but nonparadoxical. The liar is paradoxical because it asserts its own falsehood; self-reference alone is not enough ("this sentence has five words" speaks about itself without paradox). (e) Gödel's statement is not a paradox: it is true in the standard model but unprovable in the system. It is a theorem about the limits of formal proof, not a contradiction. It shows that true and provable do not always coincide.

FURTHER READING

- **Cori, R. & Lascar, D.** (2000), *Mathematical Logic: A Course With Exercises, Part I*, Oxford University Press. *English edition; propositional and predicate calculus, Boolean algebras, and completeness.*
- **Halmos, P.R.** (1960), *Naive Set Theory*, Van Nostrand (Springer reprint, 1974). *A luminous introduction, famous for its clarity.*
- **Velleman, D.J.** (2006, 2nd ed.), *How to Prove It: A Structured Approach*, Cambridge. *A remarkable pedagogical text.*
- **Hauchecorne, B.** (1988, 2nd ed. 2007), *Les contre-exemples en mathématiques, Ellipses. Useful for calibrating intuition. French-language counterexample collection retained as a complementary reading.*