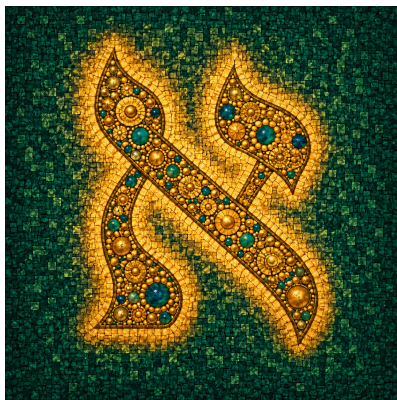




Articulated Course

in

Mathematics

I

Elements

FROM NUMBER TO FORM

William de France

CLEMENTINIUM EDITIONS

Clem

ELEMENTS

God made the integers; all else is the work of man.

LEOPOLD KRONECKER, AS REPORTED BY H. WEBER (1893)

Articulated Course in Mathematics

I

Elements

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PREFACE

This book is an invitation. It is addressed to the student who enters formal mathematics (perhaps with curiosity, perhaps with apprehension) and who discovers that behind the calculations of middle school lies an architecture. The aim is not to know everything by the end of the reading, but to begin to see: to see that $2 + 3 \times 4$ is not a sequence of arbitrary gestures but a hierarchy, that the Pythagorean theorem ties form to number, that the chain $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ tells a story of successive enlargements, each step motivated by an equation that refused to be solved in the previous setting.

This first volume teaches the first words of the mathematical language: number, distance, angle, equation, function, symmetry. It leads from school arithmetic to the threshold of university mathematics. It covers the core of secondary-school mathematics, and adds, where they illuminate the fundamental ideas, a few glimpses of the university curriculum. It is divided into seven parts: language, computation, and proof; plane geometry; geometry in space; complex numbers; functions, sequences, and analysis; probability and statistics; conics. It presupposes no prior knowledge beyond elementary arithmetic. It prepares the reader for the next volume, *Foundations*, which builds the foundations of university mathematics.

The notation follows the French original, with English conventions for intervals, decimal points, and the names of standard operations: $\mathbb{N}$ for the natural numbers, $\mathbb{R}$ for the reals, $f: A \rightarrow B$ for a map, and so on. When a notation is introduced for the first time, it is defined in the body of the article that uses it.

The reader should not worry about failing to understand everything at the first attempt. Mathematics is read slowly, pencil in hand, and it is normal to return to a passage, to redo a calculation, to let an idea ripen before coming back to it. This book is not an examination: it

is a companion for the road. May it open the way to the austere and luminous beauty of mathematical structures.

TO THE READER

This course articulates in five volumes the grammar by which reality composes itself: over fifteen hundred articles of mathematics, from the first whole number to the Langlands correspondence and from the brachistochrone to the amplituhedron. The undertaking is vast. The author does not claim it is without flaws; he asserts it is sincere.

Each article develops a single idea, from its motivation to its interpretation. Definitions are framed. Results (propositions, theorems) are marked by a bold label; in Volumes II and III, major theorems are further framed in red. Proofs are present where they illuminate, sketched where full rigor would have obscured the point, and stated without proof when their difficulty exceeds the article's scope. The text distinguishes these three cases.

Ten mathematical objects run through the collection as recurring examples: the circle S^1 , the integers $\mathbb{Z}$, the extension $\mathbb{Q}(\sqrt{2})$, the elliptic curve $y^2 = x^3 - x$, the group GL_2 , the function $\zeta(s)$, the space L^2 , the ring $K[X]$, the symmetric group $\mathfrak{S}_3$, and the torus T^2 . From trigonometric computation to the Langlands dual group, from the harmonic oscillator to Montonen-Olive duality, each of these objects reappears at every level with renewed depth. When one manifests itself, the text signals it.

The figures, numbering more than eight hundred, have been drawn with the care that a pocket format demands: every line has a reason for being, every label is clear of every curve, and the palette is limited to four colors. They do not replace proof; they precede it. The reader who looks at the figure before reading the theorem will often understand the statement before having read a word of it.

In volumes I–III, each chapter extends into graded exercises (practice exercises, method studies, expeditions), whose detailed solutions are printed upside down at the foot of each exercise: the reader may attempt the exercise without seeing the answer, then read it by turning the book

around. Whoever engages with them will find a teacher, never an examiner.

The reader may follow the linear path or take the cross-volume passages. Volumes I through III form a continuous progression from secondary school to the master's level. Volume IV ascends toward the Langlands program; Volume V, toward mathematical physics. Both assume Volume III but may be read independently of each other.

Further reading points to English editions where these can be identified, or to verified English alternatives with the same purpose. Historical attributions are retained; works available only in another language are identified as such. These readings extend the course rather than form part of its prerequisites.

Every error reported is an error corrected.

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NOTE ON THE LEVELS

The five volumes follow a progression from the first mathematical objects to questions of research:

- *Elements* (I): secondary-school mathematics, from arithmetic to the first proofs.
- *Foundations* (II): the foundations of university mathematics.
- *Graduate Mathematics* (III): deeper study and preparation for research.
- *Correspondences* (IV): the research level, from class field theory to the Langlands program.
- *Mathematical Physics* (V): the research level, from symmetries to strings.

These levels guide the reader without imposing an academic timetable. Parts and chapters mark the stages of the journey; the pace depends on the reader's background and the questions they wish to explore.

Volumes I, II, and III successively build the prerequisites for the next volume. Readers may enter at the level for which they have the necessary background. Volumes IV and V share the foundation of *Graduate Mathematics* and may be read independently of each other. The bridges at the beginning of each volume explain the change in perspective.

NOTATIONS

$\mathbb{N}$	<i>natural numbers</i>
$\mathbb{Z}$	<i>integers</i>
$\mathbb{Q}$	<i>rational numbers</i>
$\mathbb{R}$	<i>real numbers</i>
$\mathbb{C}$	<i>complex numbers</i>
$\in$	<i>belongs to</i>
$\subset, \subseteq$	<i>is a subset of (equality allowed)</i>
$\cup$	<i>union</i>
$\cap$	<i>intersection</i>
$\emptyset$	<i>empty set</i>
$\mathcal{P}(E)$	<i>power set of E</i>
$\forall$	<i>for all</i>
$\exists$	<i>there exists</i>
$\Rightarrow$	<i>implies</i>
$\Leftrightarrow$	<i>is equivalent to</i>
$ x $	<i>absolute value of x</i>
$\sqrt{x}$	<i>square root of x</i>
$[a, b], (a, b)$	<i>closed and open intervals</i>
$\leq, \geq$	<i>less than or equal, greater than or equal</i>
$\neq$	<i>is not equal to</i>
$\approx$	<i>is approximately equal to</i>
π	<i>ratio of circumference to diameter</i>
∞	<i>infinity</i>
$\sin, \cos, \tan$	<i>trigonometric functions</i>
$f'(x)$	<i>derivative of f at x</i>
$\int_a^b f(x) dx$	<i>integral of f from a to b</i>
$\sum_{k=0}^n$	<i>sum for k from 0 to n</i>
$n!$	<i>factorial of n</i>
$\binom{n}{k}$	<i>binomial coefficient</i>
i	<i>imaginary unit, $i^2 = -1$</i>
$\bar{z}$	<i>conjugate of the complex number z</i>
$ z $	<i>modulus of the complex number z</i>

$\arg(z)$	<i>argument of the complex number z</i>
$\operatorname{Re}(z), \operatorname{Im}(z)$	<i>real part, imaginary part</i>
u_n	<i>general term of a sequence</i>
$\lim_{n \rightarrow +\infty} u_n$	<i>limit of a sequence</i>
$\rightarrow$	<i>tends to</i>
$\vec{u}$	<i>vector u</i>
$\ \vec{u}\ $	<i>norm of the vector $\vec{u}$</i>
$\vec{u} \cdot \vec{v}$	<i>dot product of $\vec{u}$ and $\vec{v}$</i>
$\vec{u} \wedge \vec{v}$	<i>cross product of $\vec{u}$ and $\vec{v}$</i>
$\det(\vec{u}, \vec{v})$	<i>determinant of two vectors</i>
e	<i>Euler's number ($e \approx 2.718$)</i>
$\ln x$	<i>natural logarithm of x</i>
$e^x, \exp(x)$	<i>exponential of x</i>
$a \mid b$	<i>a divides b</i>
$a \equiv b \pmod{n}$	<i>a and b are congruent modulo n</i>
$\gcd(a, b)$	<i>greatest common divisor</i>
$\operatorname{lcm}(a, b)$	<i>least common multiple</i>
$P(A)$	<i>probability of the event A</i>
$P(A \mid B)$	<i>probability of A given B</i>
$E(X)$	<i>expectation of the random variable X</i>
$V(X)$	<i>variance of the random variable X</i>

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Part I

Language, Computation, and Proof

Computation comes first: numerical and algebraic expressions, order and absolute value, equations and inequalities. Proof follows, with sets, quantifiers, and the principal modes of reasoning. The arithmetic of the integers runs from Euclidean division to prime numbers and congruences. The study of the quadratic polynomial $ax^2 + bx + c$, of its discriminant and its vertex form, opens the first questions of optimization.

CHAPTER I

NUMERICAL AND ALGEBRAIC COMPUTATION

Every grammar begins with an alphabet. Before formulating theorems or building theories, one must learn to write: setting the order of operations, handling fractions and powers, and recognizing standard algebraic identities. These rules are not arbitrary conventions; they are the elementary syntax of a language in which reality can be described.

Articles in this chapter:

- 001. *Notation, Precedence, Numerical Computation — Why $2 + 3 \times 4$ is not 20: arithmetic is not read left to right*
- 002. *Fractions, Powers, Roots — Three families of notation that appear everywhere; mastering them is equipping oneself for what follows*
- 003. *Proportionality and the Rule of Three — Proportionality is the simplest model of a relationship between two quantities*
- 004. *Expanding and Factoring — The same expression can be written in expanded or factored form; each form has its uses*
- 005. *Standard Algebraic Identities — Expanding $(a + b)^2$ always gives the same pattern: not by chance, but by distributivity*
- 006. *The Binomial Theorem — We know $(a + b)^2$; what happens for $(a + b)^3$, $(a + b)^4$ and beyond?*
- 007. *Inequalities and Common Computational Errors — Many errors come from applying rules of equality, wrongly, to inequalities*

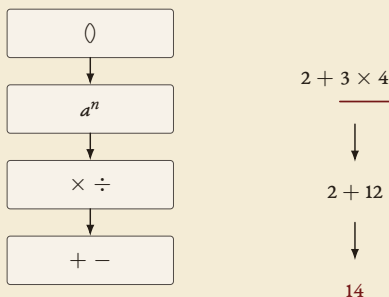
I. NOTATION, PRECEDENCE, NUMERICAL COMPUTATION.

WHAT is $2 + 3 \times 4$? Many readers, the first time, answer 20. We read left to right, add first, multiply second, and the calculation seems finished. The answer, though, is 14. The mistake is not

one of carelessness; it is one of grammar. A computation is not a sentence to be read in sequence. It is a structured expression, and that structure obeys strict rules of precedence.

Why do these rules exist? Without them, a single string of symbols would produce different numbers in different hands. Notation would become ambiguous, and ambiguous notation is useless. The order of operations is the shared convention that resolves this.

The figure below sums up this hierarchy and shows why multiplication executes before addition in $2 + 3 \times 4$.



Order of operations: the tightest-binding operations execute first

DEFINITION 1.1: Operations are performed in the following order of precedence:

1. parentheses (starting from the innermost),
2. exponents,
3. multiplications and divisions (left to right),
4. additions and subtractions (left to right).

A minus sign that marks the opposite of a number, as in -3^2 , is treated as multiplication by -1 : after exponents.

The logic is hierarchical: the tighter an operation binds its operands, the earlier it executes. Exponentiation binds more

tightly than multiplication, which binds more tightly than addition. Think of punctuation: a comma separates less sharply than a semicolon, and a semicolon less sharply than a period. The analogy is one of hierarchy, not of execution: in $2 + 3 \times 4$, the multiplication binds 3 and 4 into a unit before addition acts.

Why does multiplication come before addition, and not the reverse? Because sums of products are everywhere: a total price $3 \times 4.50 + 2 \times 1.20$, an expression $3x + 2$. With this convention they need no parentheses at all; the opposite convention would require them on every line. The rule picks the most economical notation for the most frequent expressions: it is a convention, but not a whim.

EXAMPLE: Compute $5 + 2 \times (3^2 - 1) \div 4$. Start with the parentheses; inside them, the exponent comes first: $3^2 = 9$, then the subtraction: $9 - 1 = 8$. Next the multiplication: $2 \times 8 = 16$. Then the division: $16 \div 4 = 4$. Finally the addition: $5 + 4 = 9$.

Parentheses are not decoration. They are an *instruction*: “evaluate this first, whatever the usual precedence dictates.”

EXAMPLE: Compare $(2+3) \times 4 = 20$ and $2+3 \times 4 = 14$. The symbols are nearly identical, but the parentheses change the structure, and therefore the result.

The fraction bar, too, plays a structural role that is easy to overlook: it acts as invisible parentheses around the numerator and the denominator. Writing $\frac{2+3}{4}$ means $(2+3) \div 4$, not $2+3 \div 4$.

COMMON ERROR: Confusing -3^2 with $(-3)^2$. The expression -3^2 means $-(3^2) = -9$: the exponent applies to 3 alone, and the negative sign applies afterward. The expression $(-3)^2$ means $(-3) \times (-3) = 9$: the exponent applies to the number -3 as a whole. A single placement of parentheses flips the sign of the result.

Precision in notation is the first form of mathematical rigor, and the most easily neglected. A line of reasoning can be correct in

the mind and false on the page. Respecting precedence, placing parentheses where they belong, writing each intermediate step with care: this is where all serious work begins. Not because the conventions are sacred, but because without them notation stops doing the one thing it was designed for: communicating unambiguously.

The first exercise checks that precedence is respected in a calculation with several levels: division before the addition and the subtraction. The second tests the most common trap in this chapter: confusing $-a^2$ with $(-a)^2$, which changes the sign of the result.

- Compute $4 + 6 \div 2 - 1$.

Solution: Apply the order of operations: first division, then additions and subtractions from left to right. (1) Division: $6 \div 2 = 3$. (2) The expression becomes $4 + 3 - 1$. From left to right: $4 + 3 = 7$, then $7 - 1 = 6$. Answer: 6. Common error: doing $4 + 6 + 2 - 1$ first, forgetting that division has priority.

- Without a calculator, what are -5^2 and $(-5)^2$?

Solution: For -5^2 , the exponent applies only to 5, not to the minus sign, so $-5^2 = -(5^2) = -25$. For $(-5)^2$, the exponent applies to the whole number -5 , so $(-5)^2 = (-5) \times (-5) = 25$ because a negative times a negative is positive. The two answers have opposite signs: -25 and 25 . Parentheses change the meaning.

2. FRACTIONS, POWERS, ROOTS.

CAN one add $\frac{2}{3}$ and $\frac{5}{6}$ by simply adding the tops and the bottoms?

Does $2^3 \times 2^5$ equal 2^{15} ? Is $\sqrt{9+16}$ the same as $\sqrt{9} + \sqrt{16}$? In each case the quick answer is tempting and wrong. Fractions, powers, and roots each follow their own grammar, and that grammar admits no guesswork.

These three families of notation are the building blocks of nearly every algebraic calculation. One learns their rules once; after that, they run in the background of every computation that follows. But the smallest confusion produces errors.

DEFINITION 2.1: A *fraction* is an expression $\frac{a}{b}$, where a and b are numbers (most often integers) with $b \neq 0$; a is the numerator, b the denominator. Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal, that is, they represent the same number, if and only if $a \times d = b \times c$.

RULE: To add two fractions, bring them to a common denominator. To multiply them, multiply numerators together and denominators together. To divide by a nonzero fraction, multiply by its reciprocal.

EXAMPLE: $\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6} = \frac{3}{2}$. And $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \times \frac{8}{9} = \frac{24}{36} = \frac{2}{3}$.

Adding the tops and the bottoms would give $\frac{1}{2} + \frac{1}{2} = \frac{2}{4} = \frac{1}{2}$: two halves would make one half. Parts can be added only when they have the same size, that is, the same denominator.

Addition requires common ground: a shared denominator. Multiplication, paradoxically the simpler operation, needs none. But fractions alone are not enough. The moment a calculation involves repetition, we need a notation that compresses many identical factors into a single symbol.

DEFINITION 2.2: For any real number a and any natural number $n \geq 1$, we set $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$. For $a \neq 0$, we further set, by convention, $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$.

RULE: For $a, b \neq 0$ and integers m, n , powers satisfy:

$$a^n a^m = a^{n+m}, \quad \frac{a^n}{a^m} = a^{n-m}, \quad (a^n)^m = a^{nm},$$

$$(ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Multiplying powers of the same base amounts to adding their exponents. The reason is immediate: $a^3 \times a^5$ is three copies of a multiplied by five more, giving eight in total.

The same rule explains the conventions in the definition. For $\frac{a^n}{a^n} = a^{n-n}$ to remain true, we need $a^0 = 1$; for $a^n \times a^{-n} = a^0 = 1$, we need $a^{-n} = \frac{1}{a^n}$. These are the only conventions that keep the rules valid for all integer exponents.

EXAMPLE: $2^3 \times 2^5 \div 2^4 = 2^{3+5-4} = 2^4 = 16$. And $(3^2)^3 = 3^6 = 729$.

DEFINITION 2.3: For any real number $a \geq 0$, the *square root* $\sqrt{a}$ is the unique nonnegative real number whose square equals a : $(\sqrt{a})^2 = a$. The existence of this number is taken for granted here.

RULE: For $a, b \geq 0$: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ (with $b > 0$).

But beware: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ in general.

The square root distributes over products but not over sums. This is one of the most persistent traps in computation: the impulse to split a symbol across a sum is a reflex one must learn to override.

EXAMPLE: $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$. To rationalize $\frac{1}{\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$: $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

A single thread runs through all three families: they organize multiplication, its repetition, and the operations that undo them.

The fraction $\frac{a}{b}$ undoes multiplication ($b \times \frac{a}{b} = a$). The power a^n compresses repeated multiplication into one symbol. The root $\sqrt{a}$ undoes squaring on nonnegative numbers; in general, $\sqrt{x^2} = |x|$, the absolute value of x (article 008). This architecture of operations paired with their reversals runs through the whole of algebra, from the fractions we meet here all the way to the logarithm, which will one day undo the exponential.

The article covers three families of notation, and each deserves its own short drill. The first exercise tests addition of fractions and the search for a common denominator. The second checks the exponent rules: add exponents for a product, subtract them for a quotient. The third tests simplification of a square root and the trap $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.

- Compute $\frac{2}{3} + \frac{3}{4}$.

Solution: The fractions cannot be added directly because the denominators are different: 3 and 4. Use a common denominator: the least common multiple of 3 and 4 is 12. Convert: $\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$ and $\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$. Add the numerators: $\frac{8}{12} + \frac{9}{12} = \frac{17}{12}$. Check: $\frac{17}{12} \approx 1.417$, and $\frac{2}{3} + \frac{3}{4} \approx 0.667 + 0.750 = 1.417$.

- Simplify $2^5 \times 2^3 \div 2^6$.

Solution: The base is the same, namely 2, throughout. Rule: add exponents for multiplication and subtract them for division. First $2^5 \times 2^3 = 2^{5+3} = 2^8$. Then $2^8 \div 2^6 = 2^{8-6} = 2^2 = 4$. Check: $2^5 = 32$, $2^3 = 8$, $32 \times 8 = 256$, $2^6 = 64$, and $256 \div 64 = 4$.

- Simplify $\sqrt{75}$. Then check that $\sqrt{4+9} \neq \sqrt{4} + \sqrt{9}$.

Solution: For $\sqrt{75}$, look for the largest perfect square dividing 75. Since $75 = 25 \times 3$ and $25 = 5^2$, we get $\sqrt{75} = \sqrt{25 \times 3} = 5\sqrt{3}$. Check: $(5\sqrt{3})^2 = 25 \times 3 = 75$. For the trap, $\sqrt{4+9} \approx \sqrt{13} \approx 3.606$, while $\sqrt{4} + \sqrt{9} = 2 + 3 = 5$. Thus $3.606 \neq 5$. The square root of a sum is not the sum of the square roots.

3. PROPORTIONALITY AND THE RULE OF THREE.

Two candies cost one euro, so four cost two. A child sees this without instruction: double the quantity, double the price. The reasoning feels so self-evident that one might take it for a universal law. It is not. Double the temperature of an oven and the cake does not bake in half the time. Proportionality is the simplest model relating two quantities; the real difficulty lies in knowing where it applies and where it deceives.

DEFINITION 3.1: Two quantities are *proportional* when one is obtained from the other by multiplying by the same fixed number, called the *constant of proportionality*.

EXAMPLE: If 3 kg of apples cost €4.50, the constant of proportionality is $\frac{4.50}{3} = 1.50$, that is, €1.50 per kg. The price of 7 kg is therefore $7 \times 1.50 = 10.50$, or €10.50.

The constant is the single number that binds the two quantities: once it is known, every value on one side determines the corresponding value on the other. Three values suffice to find a fourth; that is the purpose of the rule of three.

RULE: The *rule of three* (cross-multiplication) states: if $\frac{a}{b} = \frac{c}{d}$, then $a \times d = b \times c$. Given three of the four quantities, the fourth is uniquely determined whenever the required division is by a nonzero number.

EXAMPLE: If 3 kg of apples cost €4.50, how much do 5 kg cost? The quantities are proportional, so $\frac{4.50}{3} = \frac{x}{5}$. Cross-multiplying gives $3x = 5 \times 4.50 = 22.50$, so $x = 7.50$: the price is €7.50.

The rule of three assumes *direct* proportionality. The next example shows what changes when the relationship is inverse.

EXAMPLE: Assume each worker maintains the same productivity and the job can be shared without lost time. 5 workers com-

plete a job in 12 days. How many days for 8 workers? Here the proportionality is inverse: more workers, fewer days. The product $\text{workers} \times \text{days}$ is constant: $5 \times 12 = 60$, so 8 workers take $60 \div 8 = 7.5$ days.

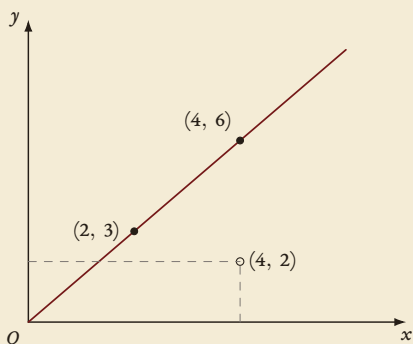
This example exposes the crucial distinction: inverse proportionality runs against the intuition of direct proportion. Increasing one factor decreases the other. The type of relationship must be identified before the method is applied.

DEFINITION 3.2: A percentage is a fraction with denominator 100. To say “15% of x ” means $\frac{15}{100} \times x = 0.15x$.

EXAMPLE: Increase a price of €120 by 15%: the new price is $120 \times 1.15 = 138$, that is, €138. Now decrease it by 15%: $138 \times 0.85 = 117.30$, that is, €117.30. The original €120 does not come back. A 15% rise followed by a 15% fall does not cancel out, because the two percentages apply to different base amounts.

This catches most people off guard. It becomes clear once a percentage is seen as a multiplication: a 15% increase multiplies by 1.15, a 15% decrease by 0.85. The two changes therefore combine by multiplication: $1.15 \times 0.85 = 0.9775$, an overall decrease of 2.25%. To return to the starting price, one would have to divide by 1.15, that is, decrease by about 13.04%. The arithmetic is straightforward, yet the insight extends to compound interest, inflation adjustments, and every setting where successive percentage changes accumulate.

Direct proportionality (the quotient y/x stays constant) and inverse proportionality (the product xy stays constant) must be carefully distinguished. The direction of variation does not settle the matter: for $x > 0$, $y = x^2$ increases with x without being proportional to x . Graphically, direct proportionality has a clean signature: the data points lie on a straight line through the origin.



Direct proportionality: points aligned on a line through the origin

The figure confirms it: proportional pairs fall on a line through the origin; any point off that line signals a departure from the model.

Speed, scale on a map, chemical concentration, currency exchange: proportionality is the first mathematical model one encounters, and often the last one reaches for when the situation grows complex. Learning to recognize it matters; learning to reject it when it fails matters more. That discrimination, between a pattern that holds and one that merely resembles, is the starting point of all serious modeling.

The first exercise tests inverse proportionality: the trap is that more workers mean fewer days, not more. The second checks why an increase followed by a decrease by the same percentage does not return to the starting point.

- 8 workers complete a job in 6 days. Assuming constant individual productivity and no time lost in sharing the job, how many days would 12 workers need?

Solution: This is inverse proportionality: more workers mean fewer days. The product workers $\times$ days is constant: $8 \times 6 = 48$. Thus 12 workers need $48 \div 12 = 4$ days. Check: $12 \times 4 = 48 = 8 \times 6$. Common error: applying direct proportionality, $\frac{8}{12} \times 6 = 9$ days, which would give more days for more workers.

- An item costs €250. Its price is increased by 20%, then reduced by 20%. Is the final price €250?

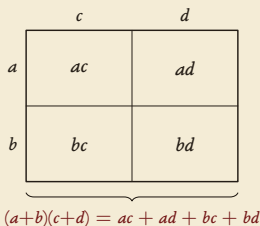
Solution: Increase: $250 \times 1.20 = 300$, that is, €300. Reduction: $300 \times 0.80 = 240$, that is, €240. The final price is not €250 but €240. Why? The 20% increase is taken from 250 and adds 50, but the 20% decrease is taken from 300 and subtracts 60. The two percentages do not represent the same amount. In general, multiplying by 1.20 and then by 0.80 gives $1.20 \times 0.80 = 0.96$, a total decrease of 4%.

4. EXPANDING AND FACTORING.

LOOK at the expression $2x^2 - 7x - 15$. As it stands, it tells us little at a glance. Now write the same expression as $(2x + 3)(x - 5)$: immediately we read off the values that make it vanish, called its *roots*: $x = -\frac{3}{2}$ and $x = 5$. The numbers have not changed; only the *form* has, and with it what we can see at a glance. An algebraic expression, like a sentence, can be rephrased without altering its content. The expanded form lays the terms out; the factored form presses them into a product. Each illuminates something the other hides.

DEFINITION 4.1: *Expanding* means transforming a product into a sum. *Factoring* means transforming a sum into a product. They are the two directions in which one equality can be read.

RULE: Simple distributivity: $k(a + b) = ka + kb$. Double distributivity: $(a + b)(c + d) = ac + ad + bc + bd$.



Double distributivity: each cell contains a partial product

EXAMPLE: Expand $(2x + 3)(x - 5)$. By double distributivity: $2x \cdot x + 2x \cdot (-5) + 3 \cdot x + 3 \cdot (-5) = 2x^2 - 10x + 3x - 15 = 2x^2 - 7x - 15$.

Expanding is mechanical: multiply every term of the first factor by every term of the second, then collect like terms. Factoring is the harder move. It requires seeing a hidden common factor, recognizing a structure where only a sum of terms is visible.

RULE: To factor out a common factor, identify the greatest common factor of all terms and place it in front.

EXAMPLE: Factor $6x^3 - 9x^2$. The common factor is $3x^2$: $6x^3 - 9x^2 = 3x^2(2x - 3)$.

A second resource comes in the next article: recognizing a standard algebraic identity, a perfect square or a difference of squares. Which form to use depends entirely on the problem at hand. To solve an equation of the type “product equals zero,” we need the factored form: $A \times B = 0$ if and only if $A = 0$ or $B = 0$ (a property of the real numbers, taken for granted here). To simplify or compare expressions, the expanded and collected form is often more convenient. The choice is strategic: which representation makes the problem transparent? This is a lesson that reaches well beyond algebraic computation. In mathematics, the difficulty of a problem often depends less on its substance than on the angle from which we approach it.

Expanding and factoring are two readings of a single reality: one spreads out, the other condenses. This back-and-forth between two forms of the same expression is the first instance of a gesture that pervades all of mathematics: shifting the viewpoint to reveal what was, a moment ago, invisible.

Both exercises test factoring out a common factor: find the greatest common factor, and, in the second, do not forget the term that becomes 1.

- Factor $12x^2 - 8x$.

Solution: Look for the greatest common factor of the two terms. For the coefficients, $\gcd(12, 8) = 4$. For the variables, x^2 and x have x in common, with the smaller exponent. Thus the common factor is $4x$. Factor: $12x^2 - 8x = 4x(3x - 2)$. Check by expanding: $4x \cdot 3x = 12x^2$ and $4x \cdot (-2) = -8x$.

- Factor $15x^3 - 10x^2 + 5x$.

Solution: For the coefficients, $\gcd(15, 10, 5) = 5$. For the variables, x^3 , x^2 , and x have x in common. Thus the common factor is $5x$: $15x^3 - 10x^2 + 5x = 5x(3x^2 - 2x + 1)$. Check by expanding: $5x \cdot 3x^2 = 15x^3$, $5x \cdot (-2x) = -10x^2$, $5x \cdot 1 = 5x$. Common error: writing $5x(3x^2 - 2x)$, forgetting that the last term $5x$ becomes 1, not 0.

5. **STANDARD ALGEBRAIC IDENTITIES.**

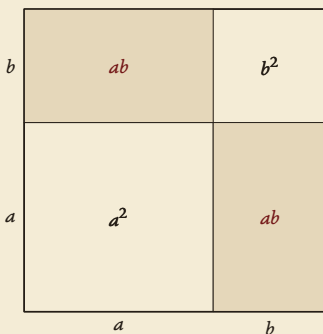
EXPAND $(2 + 3)^2 = (2 + 3)(2 + 3)$ term by term, then $(10 + 1)^2$, then $(a + b)^2$. Each time the same pattern appears: a square, another square, and a doubled product between them. This regularity is not a coincidence. It follows directly from double distributivity, seen in the previous article, and it condenses into three formulas so ubiquitous that they have earned their own name: the standard algebraic identities.

PROPOSITION 5.1: For all real numbers a and b :

$$(a + b)^2 = a^2 + 2ab + b^2$$

ANNOTATED PROOF: We expand by distributivity: $(a + b)^2 = (a + b)(a + b) = a \cdot a + a \cdot b + b \cdot a + b \cdot b = a^2 + 2ab + b^2$.

The figure below gives a proof without words. For $a, b \geq 0$, a square of side $a + b$ splits into four regions whose areas are a^2 , ab , ab , and b^2 ; the algebraic computation above holds for all reals. Algebra and geometry say the same thing; the two shaded rectangles, each of area ab , account for the coefficient 2.



Geometric proof: the square of side $a + b$ decomposes into four regions

PROPOSITION 5.2: For all real numbers a and b :

$$(a - b)^2 = a^2 - 2ab + b^2$$

EXAMPLE: Compute $(3x - 2)^2$. With $a = 3x$ and $b = 2$: $(3x - 2)^2 = (3x)^2 - 2 \cdot 3x \cdot 2 + 2^2 = 9x^2 - 12x + 4$.

PROPOSITION 5.3: For all real numbers a and b :

$$(a + b)(a - b) = a^2 - b^2$$

This third identity is perhaps the most elegant. It says that the product of two numbers equidistant from a midpoint falls just short of the square of that midpoint; the shortfall is the square of the offset.

EXAMPLE: *How much is 101×99 ? Write $101 \times 99 = (100 + 1)(100 - 1) = 100^2 - 1^2 = 10\,000 - 1 = 9\,999$. An identity turns a product into mental arithmetic.*

PROPOSITION 5.4: Identities of degree three. For all reals a and b :

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

The first two come from expanding $(a \pm b)^2(a \pm b)$, the last two from expanding the right-hand side. They extend the second-degree identities and come into play as soon as a cube appears.

These identities work in both directions. Turning a product into a sum is *expanding*; turning a sum into a product is *factoring*. The strength of these identities lies in this reversibility; the direction one chooses depends on what the problem requires.

EXAMPLE: *The expression $x^2 - 6x + 9$ can be recognized as $x^2 - 2 \cdot x \cdot 3 + 3^2 = (x - 3)^2$. Likewise, $4x^2 - 25 = (2x)^2 - 5^2 = (2x - 5)(2x + 5)$.*

Three formulas of degree two, four of degree three, two directions, applications beyond counting. From mental arithmetic to equation-solving, from factoring to the study of functions, these identities run through the whole of secondary mathematics and well beyond it. They form a core vocabulary that one uses constantly, often without remembering that it was once learned.

The first exercise practices the identity $(a-b)^2$ in a case with coefficients: identify $a = 5x$ and $b = 3$, and do not forget the double product. The second uses $(a+b)(a-b)$ for mental arithmetic by turning a product of two numbers near a pivot into a square minus a square.

- Expand $(5x - 3)^2$.

Solution: Identify $a = 5x$ and $b = 3$. The formula $(a-b)^2 = a^2 - 2ab + b^2$ gives $(5x)^2 - 2 \times 5x \times 3 + 3^2 = 25x^2 - 30x + 9$. Detail: $(5x)^2 = 25x^2$, so the coefficient is squared as well. The double product is $2 \times 5x \times 3 = 30x$, and $3^2 = 9$. Answer: $25x^2 - 30x + 9$.

- Compute 52×48 mentally.

Solution: Recognize $(50+2)(50-2)$. This is the identity $(a+b)(a-b) = a^2 - b^2$ with $a = 50$ and $b = 2$. Thus $52 \times 48 = 50^2 - 2^2 = 2500 - 4 = 2496$. Check: $52 \times 48 = 52 \times 50 - 52 \times 2 = 2600 - 104 = 2496$. A standard identity turns the product into easy mental arithmetic.

6. THE BINOMIAL THEOREM.

WE know how to expand $(a+b)^2$. With some patience, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ can be worked out by hand. But $(a+b)^7$? $(a+b)^{20}$? Applying distributivity term by term quickly becomes impractical. We need a general formula, and one exists: the binomial theorem. It rests on a triangle of numbers whose apparent simplicity conceals a striking depth. The formula for whole-number exponents and the triangle are old: they appear with al-Karaji around the year 1000, with Yang Hui in the thirteenth century, and with Pascal in 1654. Newton extended the formula to fractional exponents around 1665, which is why French textbooks call it Newton's binomial.

Before we can state the formula, we need its basic ingredient: a number that counts how many ways one can select k objects from a collection of n .

DEFINITION 6.1: For integers $n \geq 0$ and $0 \leq k \leq n$, the *binomial coefficient* is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $n! = 1 \times 2 \times \cdots \times n$ (with $0! = 1$).

We will prove below that this number counts the ways to choose k elements from n .

EXAMPLE: For $n = 4$: $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$, $\binom{4}{4} = 1$.

The values form a palindrome. This is no accident: exchanging k and $n - k$ leaves the formula unchanged, and once we know that $\binom{n}{k}$ counts choices, choosing k elements from n will be the same as deciding which $n - k$ elements to leave out.

PROPOSITION 6.2: *Pascal's triangle.* For $n \geq 2$ and $1 \leq k \leq n - 1$, that is, except for the 1s at the two ends of each row, each binomial coefficient is the sum of the two directly above it:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

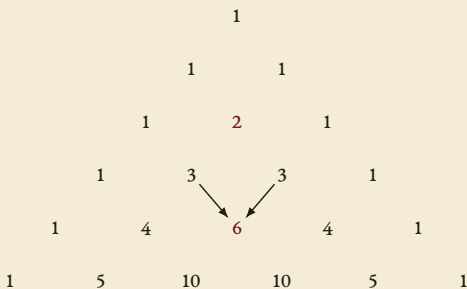
ANNOTATED PROOF: Bring the two fractions to the common denominator $k!(n-k)!$ by multiplying the first by k and the second by $n - k$, top and bottom:

$$\begin{aligned} \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} &= \frac{(n-1)!(k + (n-k))}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!}. \end{aligned}$$

PROPOSITION 6.3: For $0 \leq k \leq n$, the number of ways to choose k elements from n is $\binom{n}{k}$.

ANNOTATED PROOF: Call this number $N(n, k)$. There is one way to choose nothing and one way to take everything: $N(n, 0) = N(n, n) = 1$. For $1 \leq k \leq n - 1$, set one of the n elements aside. A choice of k elements either contains it, and then $k - 1$ elements remain to be taken from the other $n - 1$, or does not contain it, and then all k elements are taken from the other $n - 1$: $N(n, k) = N(n - 1, k - 1) + N(n - 1, k)$. So the numbers $N(n, k)$ take the same values as the $\binom{n}{k}$ at the ends of each row and are built by the same rule; row after row, they are equal to them.

This recurrence builds the coefficients row by row, like a wall of bricks: each entry is the sum of the two entries just above it. The first few rows read 1; 1 1; 1 2 1; 1 3 3 1; 1 4 6 4 1; and so on.



Pascal's triangle: each coefficient is the sum of the two above

The diagram makes the construction visible: each entry is the sum of its two parents, and the symmetry of each row reflects the identity $\binom{n}{k} = \binom{n}{n-k}$.

PROPOSITION 6.4: *The binomial theorem.* For all real numbers a, b and every integer $n \geq 1$, with the convention $x^0 = 1$ for every real x (including $x = 0$) inside this formula:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

The symbol $\sum$ abbreviates an indexed sum: $\sum_{k=0}^n t_k = t_0 + t_1 + \cdots + t_n$. Indexed sums will be practiced in their own right in chapter 15.

ANNOTATED PROOF: Expanding $(a + b)^n = (a + b)(a + b) \cdots (a + b)$ means choosing either a or b in each of the n factors, then adding all the products obtained. The product $a^{n-k} b^k$ appears each time b is chosen in exactly k factors: as many times as there are ways to choose those k factors from n , that is, $\binom{n}{k}$ times by the previous proposition. After collecting terms, the expansion has $n + 1$ terms.

EXAMPLE: Expand $(x + 2)^4$. The row $n = 4$ gives coefficients 1, 4, 6, 4, 1. So $(x + 2)^4 = x^4 + 4x^3 \cdot 2 + 6x^2 \cdot 4 + 4x \cdot 8 + 16 = x^4 + 8x^3 + 24x^2 + 32x + 16$.

When $a = 1$, the coefficients of $(1 + x)^n$ are directly the $\binom{n}{k}$.

EXAMPLE: $(1 - x)^3 = 1 - 3x + 3x^2 - x^3$. The signs alternate because $b = -x$.

The binomial theorem is more than a computation shortcut. It is a first bridge between algebra and combinatorics: between formal identities and the act of counting. We will meet it again in probability, in Taylor expansions in the next volume, and in every setting where counting and computing advance hand in hand.

The first exercise builds a row of Pascal's triangle by hand, adding adjacent coefficients. The second applies the binomial formula in a concrete case with negative b , which makes the signs alternate.

- Write the row $n = 5$ of Pascal's triangle.

Solution: The row $n = 4$ is $1, 4, 6, 4, 1$. To build the row $n = 5$, add adjacent pairs and put 1 at the ends: $1, 1 + 4 = 5, 4 + 6 = 10, 6 + 4 = 10, 4 + 1 = 5, 1$. Answer: $1, 5, 10, 10, 5, 1$. Check: $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$.

- Expand $(x - 1)^4$.

Solution: Step 1: coefficients. The row $n = 4$ of Pascal's triangle is $1, 4, 6, 4, 1$. Step 2: identify $a = x$ and $b = -1$. Step 3: powers of b : $(-1)^0 = 1, (-1)^1 = -1, (-1)^2 = 1, (-1)^3 = -1, (-1)^4 = 1$. The signs alternate. Step 4: assemble the terms. For $k = 0$: $1 \cdot x^4 \cdot 1 = x^4$. For $k = 1$: $4 \cdot x^3 \cdot (-1) = -4x^3$. For $k = 2$: $6 \cdot x^2 \cdot 1 = 6x^2$. For $k = 3$: $4 \cdot x \cdot (-1) = -4x$. For $k = 4$: $1 \cdot 1 \cdot 1 = 1$. Result: $x^4 - 4x^3 + 6x^2 - 4x + 1$. Check at $x = 2$: $2^4 = 16$ and $16 - 32 + 24 - 8 + 1 = 1$.

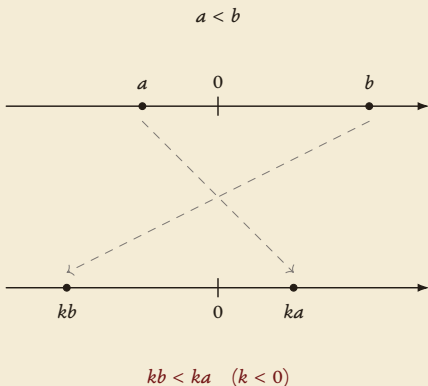
7. INEQUALITIES AND COMMON COMPUTATIONAL ERRORS.

CONSIDER the following chain of reasoning: “ $-3 < 1$, so $(-3)^2 < 1^2$, so $9 < 1$.” The conclusion is absurd, yet each step looked plausible. The error is subtle: squaring was applied to an inequality, an operation that preserves order only under certain conditions. Inequalities resemble equalities in appearance; they obey, however, a more fragile logic. That fragility is the source of some of the most stubborn errors in secondary mathematics.

RULE: Adding the same number to both sides of an inequality preserves its direction: if $a < b$, then $a + c < b + c$. Multiplying both sides by a *positive* number preserves the direction: if $a < b$ and $k > 0$, then $ka < kb$.

RULE: Multiplying both sides by a *negative* number **reverses the direction** of the inequality: if $a < b$ and $k < 0$, then $ka > kb$.

This reversal is the central danger. Addition is safe: it shifts both sides without disturbing their relative position. Multiplication by a negative number, by contrast, acts as a mirror on the number line; it flips left and right, and the ordering flips with them.



Multiplication by $k < 0$: the ordering reverses as in a mirror

EXAMPLE: Solve $-2x > 6$. Divide by -2 , which is negative: the direction reverses. This gives $x < -3$. Check: $x = -4$ works, since $-2 \times (-4) = 8 > 6$; $x = 0$ does not, since $0 > 6$ is false.

The three errors below are variations on a single theme: applying to an inequality an operation that does not preserve order.

COMMON ERROR: “If $a < b$, then $a^2 < b^2$.” This is false in general.

COUNTEREXAMPLE: Take $a = -3$ and $b = 1$. Indeed $-3 < 1$, but $(-3)^2 = 9 > 1 = 1^2$.

Squaring preserves order when both numbers are nonnegative, but can reverse it as soon as one of them is negative.

The same trap, wearing a different disguise: for two nonzero numbers, taking reciprocals reverses order when both sides share the same sign, but preserves it when the signs differ.

COMMON ERROR: “If $a < b$, then $\frac{1}{a} > \frac{1}{b}$.” This holds only when a and b have the same sign.

COUNTEREXAMPLE: Compare $\frac{1}{-2}$ and $\frac{1}{3}$. We have $-2 < 3$, yet $\frac{1}{-2} = -0.5 < 0.33 \dots = \frac{1}{3}$. The inequality between the reciprocals goes in the same direction here, because the numbers have opposite signs.

The square root lays a different kind of trap: it does not disturb ordering (it preserves order on nonnegative numbers), but it is not additive. It distributes over products, but not over sums: $\sqrt{a+b}$ and $\sqrt{a} + \sqrt{b}$ agree only when one of the two numbers is zero.

COMMON ERROR: “ $\sqrt{a^2 + b^2} = a + b$.” This is almost always false.

COUNTEREXAMPLE: With $a = 3$ and $b = 4$: $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$, while $3 + 4 = 7$. We have $5 \neq 7$.

The square root of a sum is not the sum of the square roots.

The common thread is always the same: an operation that works unconditionally on equalities demands hypotheses the moment it is applied to an inequality. Equality is preserved by every operation defined on both sides; keeping an equivalence additionally requires the operation to be reversible on the domain considered. Order is fragile; it withstands only those operations that respect its direction. This asymmetry is the reason that working with inequalities is, in a genuine sense, harder than working with equalities, and why errors concentrate there.

Knowing the traps is already half of avoiding them; but only half. The other half requires understanding *why* a trap works: which hypothesis was missing, which condition was silently assumed. In mathematics, vigilance is not a character trait. It is a skill, and one that can be trained.

The article presents three common errors, each worth a drill. The first exercise practices reversing the inequality when dividing by a negative

number. The second takes reciprocals of two positive numbers. The third tests the trap $\sqrt{a^2 + b^2} \neq a + b$.

- Solve $-3x \leq 9$.

Solution: Divide both sides by -3 . Since -3 is negative, the direction of the inequality reverses. Thus $-3x \leq 9$ gives $x \geq 9 \div (-3)$, hence $x \geq -3$. Check: for $x = -3$, $-3(-3) = 9 \leq 9$. For $x = 0$, $-3 \cdot 0 = 0 \leq 9$. For $x = -4$, $-3(-4) = 12 \leq 9$ is false. Thus $x = -4$ is not a solution, confirming $x \geq -3$.

- If $0 < a < b$, compare $\frac{1}{a}$ and $\frac{1}{b}$.

Solution: Since $a < 0$ and $b < 0$, the product ab is positive, and so is $\frac{1}{ab}$. Multiply both sides of $a < b$ by $\frac{1}{ab}$ without changing the direction: $\frac{a}{ab} < \frac{b}{ab}$, that is, $\frac{1}{b} < \frac{1}{a}$. Example: $2 < 5$ and $\frac{2}{10} < \frac{5}{10}$. For two positive numbers, taking reciprocals reverses the order.

- True or false: $\sqrt{5^2 + 12^2} = 5 + 12 = 17$?

Solution: False. $\sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$, while $5 + 12 = 17$. Thus $13 \neq 17$. The square root of a sum is not the sum of the square roots: in general, $\sqrt{a^2 + b^2} \neq a + b$. Equality holds only if $b = 0$ with $a \geq 0$, or $a = 0$ with $b \geq 0$. Here $\sqrt{5^2 + 12^2} = 13$ gives the Pythagorean triple $(5, 12, 13)$, the best known after $(3, 4, 5)$.

METHODS

Study 1. ★★ METHOD: decode a calculation involving fractions, powers, and order of operations

Real calculations often combine fractions, powers, and parentheses in a single expression. The method is unchanged: work from the deepest level to the outermost one. The difficulty lies in the layering: one must apply the rules for fractions, powers, and order of operations at the same time. The most frequent trap is forgetting that a fraction bar acts like invisible parentheses: the numerator and the denominator are each a block.

Compute $\frac{2^3 + 3 \times 5}{7 - \left(\frac{1}{2}\right)^2}$.

- Identify the numerator and the denominator. Are they separate blocks?
- Compute the numerator: first 2^3 , then 3×5 , then the addition.
- Compute the denominator: first $\left(\frac{1}{2}\right)^2$, then the subtraction.
- Carry out the division, numerator divided by denominator. Simplify if possible.
- Summarize: when a calculation combines fractions and powers, what is the method?

Solution 1: (a) The fraction bar separates two blocks: numerator = $2^3 + 3 \times 5$ and denominator = $7 - (1/2)^2$. Each is computed independently before the division. (b) Numerator: $2^3 = 8$, $3 \times 5 = 15$, and $8 + 15 = 23$. (c) Denominator: $(1/2)^2 = 1/4$, so $7 - 1/4 = 28/4 - 1/4 = 27/4$. (d) $23 \times \frac{4}{27} = 23 \times \frac{27}{27} \times \frac{4}{27} = 23 \times \frac{4}{27} \approx 3.41$. (e) Treat numerator and denominator as separate blocks, each with its internal order of operations, then divide.

Study 2. ★ METHOD: reduce to a common denominator

To add two fractions $\frac{a}{b} + \frac{c}{d}$, one needs a common denominator. The simplest choice is bd , but it is sometimes unnecessarily large. The better choice is the least common multiple of b and d : the smallest positive integer that is a multiple of both (studied in Chapter 4). Then multiply each fraction by what is needed to reach that denominator, add or subtract the numerators, and simplify if possible.

Compute $\frac{5}{6} - \frac{3}{8}$.

- What is the least common multiple of 6 and 8?

Hint: List the multiples of 8: 8, 16, 24, 32, ... Which is the first one that is also a multiple of 6?

- Convert each fraction to denominator 24.
- Carry out the subtraction.
- Is the resulting fraction reducible?
- Summarize: what is the method for adding or subtracting fractions?

Solution 2: (a) $\text{lcm}(6, 8) = 24$. (b) $\frac{6}{5} \times \frac{4}{4} = \frac{24}{20}$ and $\frac{8}{3} = \frac{8}{3} \times \frac{8}{8} = \frac{64}{24}$. (c) $\frac{24}{9} = \frac{8}{3}$. (d) $\frac{24}{11} \cdot \frac{24}{11} = \frac{576}{121}$. (e) Find the LCM of the denominators, convert both fractions, add and subtract the numerators, and simplify.

Study 3. ★★ METHOD: *identify the type of proportionality*

In a word problem, first ask: does the quotient of the quantities stay constant (direct proportionality), or does their product stay constant (inverse proportionality)? The direction of change is only a hint; a constant quotient or product determines the model. With direct proportionality, divide and then multiply: this is the rule of three. With inverse proportionality, the product of the two quantities is constant: compute that product, then divide.

Assume a constant printing rate, constant individual productivity, and a job that can be shared without lost time. Two problems. Problem A: a printer prints 120 pages in 8 minutes. How many pages in 15 minutes? Problem B: 6 workers paint a fence in 10 hours. How many hours would 15 workers need?

- (a) *Problem A: more time, more pages? Direct or inverse proportionality?*
- (b) *Compute the rate in pages per minute and deduce the number of pages in 15 minutes.*
- (c) *Problem B: more workers, more hours or fewer hours? Direct or inverse?*
- (d) *In inverse proportionality, the product is constant. Compute workers $\times$ hours, then deduce the time for 15 workers.*
- (e) *Summarize: how does one distinguish direct from inverse proportionality, and which method applies in each case?*

Solution 3: (a) Direct: more time gives more pages. (b) $120 \div 8 = 15$ pages per minute. In 15 minutes, this gives $15 \times 15 = 225$ pages. (c) Inverse: more workers give less time. The product is constant: $6 \times 10 = 60$. For 15 workers, the time is $60 \div 15 = 4$ hours. Check: $15 \times 4 = 60 = 6 \times 10$. (d) Direct: constant coefficient ($y = kx$). Inverse: constant product ($xy = c$). The direction of variation is only a hint; the constant quotient or product is what characterizes the relationship.

Study 4. ★★ METHOD: *factor an expression systematically*

To factor an expression, follow a systematic order. (1) Look for a common factor in all terms and factor it out. (2) If what remains has two terms, check for a difference of squares $a^2 - b^2$; if it has three, check for a perfect square $a^2 \pm 2ab + b^2$. (3) If nothing works, try grouping: collect the terms in pairs so that a common factor appears, as in $x^3 + x^2 + x + 1 = x^2(x+1) + (x+1) = (x^2+1)(x+1)$. The most frequent error is skipping step (1) and searching immediately for an identity.

Factor $2x^3 - 18x$.

- Is there a common factor in all terms? Which one?
- Factor it out. What remains inside the parentheses?
- Is the expression inside the parentheses a difference of squares?
- Factor completely.
- Summarize: what order should be followed when factoring?

Solution 4: (a) Yes: $2x$ is a common factor, since $2x^3 = 2x \cdot x^2$ and $18x = 2x \cdot 9$. (b) $2x^3 - 18x = 2x(x^2 - 9)$. (c) Yes: $x^2 - 9 = x^2 - 3^2$, a difference of squares with $a = x$ and $b = 3$. (d) $2x(x^2 - 9) = 2x(x+3)(x-3)$. Check: $2x \cdot x \cdot x + 2x \cdot x \cdot (-3) = 2x^3 - 6x^2$; $2x \cdot x \cdot 3 + 2x \cdot (-3) \cdot (-3) = 6x^2 - 18x$. The middle terms cancel. (e) Order: common factor first, then standard identity, then grouping.

Study 5. ★★ METHOD: *build and use Pascal's triangle*

To expand $(a + b)^n$, one needs binomial coefficients. Pascal's triangle provides them without factorials: start with 1 at the top, each number is the sum of the two above it, and the edges are always 1. The row n gives the coefficients of $(a + b)^n$. The powers of a decrease from n to 0, while the powers of b increase from 0 to n .

Expand $(2x + 3)^3$.

- Write the row $n = 3$ of Pascal's triangle.
Hint: Row $n = 2$ is 1, 2, 1. For $n = 3$: 1, 1 + 2, 2 + 1, 1.
- Identify $a = 2x$ and $b = 3$. Write the powers of a and b for $k = 0, 1, 2, 3$.
- Assemble: for each k , multiply the coefficient by $a^{3-k} \times b^k$.
- Simplify each term and add them.

(e) Summarize: how does one expand $(a + b)^n$ using Pascal's triangle?

Solution 6: (a) Row $n = 3$: 1, 3, 3, 1. (b) Powers of $a = 2x$: $(2x)^3 = 8x^3$, $(2x)^2 = 4x^2$, $(2x)^1 = 2x$, $(2x)^0 = 1$. Powers of $b = 3$: $3^0 = 1$, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$. (c) Term by term: for $k = 0$, $1 \times 8x^3 \times 1 = 8x^3$; for $k = 1$, $3 \times 4x^2 \times 3 = 36x^2$; for $k = 2$, $3 \times 2x \times 9 = 54x$; for $k = 3$, $1 \times 1 \times 27 = 27$. (d) Add: $8x^3 + 36x^2 + 54x + 27$. Check at $x = 1$: $(2 + 3)^3 = 125$ and $8 + 36 + 54 + 27 = 125$. (e) Build row n , identify a and b , compute each term separately, the coefficient multiplied by $a^{n-k} \times b^k$, then add.

Study 6. ★★. trap METHOD: *manipulate inequalities without making sign errors*

Inequalities are manipulated like equalities, except when multiplying or dividing by a negative number: then the direction reverses. The classic error is forgetting this reversal. Squaring preserves order on nonnegative numbers; with negative inputs, the direction needs a separate check. The method is to ask, at each operation: does this operation preserve the order?

Solve $\frac{2 - x}{3} > 1$.

(a) Multiply both sides by 3. Does the direction change?

Hint: Is the number 3 positive or negative?

(b) You obtain $2 - x > 3$. Isolate x .

(c) At which step did the direction change, if at all?

(d) Check your answer with $x = -2$, which should work, and $x = 0$, which should not.

(e) Summarize: which operations reverse the direction of an inequality?

Solution 6: (a) Since $3 > 0$, there is no reversal: $2 - x > 3$. (b) $2 - x > 3$ gives $-x > 1$. Multiplying by -1 reverses the direction: $x < -1$. (c) The direction changed when multiplying by -1 . (d) For $x = -2$, $\frac{3}{2 - (-2)} = \frac{3}{4} \approx 1.33 > 1$, so it works. For $x = 0$, $\frac{3}{2 - 0} = \frac{3}{2} \approx 0.67 > 1$ is false. Correct: $0 \not< -1$. (e) Multiplying or dividing by a negative number reverses the direction. Adding or subtracting never does.

METHOD RECAP

ORDER OF OPERATIONS: *Work from the deepest level, parentheses, to the outermost level, addition. Reading left to right only decides between operations of the same level: multiplications and divisions among themselves, additions and subtractions among themselves.*

COMMON DENOMINATOR: *Find the LCM of the denominators, convert each fraction, add or subtract the numerators, and simplify.*

PROPORTIONALITY: *Direct means constant coefficient ($y = kx$). Inverse means constant product ($xy = C$). The direction of variation is only a hint; only the constant quotient or product characterizes the relationship.*

FACTORING: *Order: (1) common factor, (2) standard identity, (3) grouping. Never skip step (1).*

PASCAL'S TRIANGLE: *Build it row by row: each number is the sum of the two above it. Row n gives the coefficients of $(a + b)^n$.*

INEQUALITIES: *At each operation, ask whether the direction changes. Among the operations used on a linear inequality, only multiplying or dividing by a negative number reverses it; other moves (taking reciprocals, squaring) require a monotonicity check.*

EXPEDITIONS**EXPEDITION 1: MENTAL ARITHMETIC AND IDENTITIES**

Prerequisite: study 4.

Can one multiply 97×103 mentally? What about 998×1002 ? Standard algebraic identities turn complicated products into mental arithmetic.

(a) *Compute 97×103 using the identity $(a + b)(a - b) = a^2 - b^2$.*

Hint: $97 = 100 - 3$ and $103 = 100 + 3$.

- (b) Compute 998×1002 in the same way.
- (c) Compute 47×53 and 196×204 .
- (d) Invent three products of your own and compute them mentally.
- (e) Why does this method work so well when the two numbers are symmetric around a round number?
- (f) Now try 99^2 . Which identity should be used?

Hint: $99 = 100 - 1$, so $99^2 = (100 - 1)^2 = \dots$

The identity $(a + b)(a - b) = a^2 - b^2$ turns a product of two numbers symmetric around a pivot into a square minus a small square. And $(a - b)^2 = a^2 - 2ab + b^2$ computes squares close to a round number. Standard algebraic identities are not just formulas to memorize; they are tools for mental arithmetic.

Report 1: (a) $97 \times 103 = (100 - 3)(100 + 3) = 10000 - 9 = 9991$. (b) $966 \times 1002 = 1000 \times 966 + 2 \times 966 = 966000 + 1932 = 967932$. (c) $47 \times 53 = 2500 + 6 - 4 = 2502$. (d) $196 \times 204 = 40000 + 16 - 3984 = 36016$. (e) Because a^2 is an easy round square and b^2 is small. (f) $99^2 = (100 - 1)^2 = 10000 - 200 + 1 = 9801$.

EXPEDITION 2: THE SECRETS OF PASCAL'S TRIANGLE

Prerequisite: study 5.

Pascal's triangle looks simple: 1s on the edges, and each number is the sum of the two above it. But this triangle hides remarkable patterns.

- (a) Build the first 8 rows of the triangle, from $n = 0$ to $n = 7$.
- (b) Compute the sum of the numbers in each row. What do you notice?

Hint: Row 0: 1. Row 1: $1 + 1 = 2$. Row 2: $1 + 2 + 1 = 4$. Continue.

- (c) Look at the second diagonal, the one beginning 1, 2, 3, 4. Which sequence do you recognize?
- (d) Look at the third diagonal, the one beginning 1, 3, 6, 10. What are these numbers called?

Hint: They are numbers of the form $1 + 2 + 3 + \dots + k$. They are called triangular numbers.

- (e) Add the rising diagonals. To find them, start from the leftmost 1 in each row and move diagonally upward to the right, one row at each step. First diagonal: the top 1 alone. Second: the left 1 in row $n = 1$

alone. Third: the left 1 in row $n = 2$ and the right 1 in row $n = 1$, so $1 + 1 = 2$. Continue. Which sequence appears?

Hint: Diag. 1: 1. Diag. 2: 1. Diag. 3: $1 + 1 = 2$. Diag. 4: $1 + 2 = 3$. Diag. 5: $1 + 3 + 1 = 5$. Diag. 6: $1 + 4 + 3 = 8$. Do you recognize 1, 1, 2, 3, 5, 8, 13, ...?

(f) Why does the sum of row n equal 2^n ?

Hint: $(1 + 1)^n = \sum \binom{n}{k} 1^{n-k} 1^k = \sum \binom{n}{k}$.

Pascal's triangle contains the positive integers, the triangular numbers, the Fibonacci sequence, and the powers of 2. A triangle of numbers built by such a simple rule produces striking combinatorial richness. It will return in probability, through the binomial distribution; in analysis, through expansions; and in number theory.

Report 2: (b) Sums: 1, 2, 4, 8, 16, 32, 64, 128, ... (c) The positive integers: 1, 2, 3, 4, 5, ... (d) The triangular numbers: 1, 3, 6, 10, 15, 21, 28, ... (e) The Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, ... (f) Because $(1 + 1)^n = 2^n$, and expanding $(1 + 1)^n$ by the binomial theorem gives the sum of the coefficients $\sum \binom{n}{k}$.

EXPEDITION 3: WHY $(1.01)^n$ GROWS SO FAST

Prerequisites: studies 3, 5.

An increase of 1% per year seems tiny. But after 100 years?

(a) Compute $(1.01)^2$ by hand using $(1 + x)^2 = 1 + 2x + x^2$ with $x = 0.01$.

(b) Compute $(1.01)^{10}$ with a calculator. Compare it with $1 + 10 \times 0.01 = 1.10$.

(c) Compute $(1.01)^{100}$. Compare it with $1 + 100 \times 0.01 = 2$.

Hint: Use the power key of the calculator, then compare the two results.

(d) With a calculator, compute $(1.01)^{365}$, one year of daily growth at 1%. The result may surprise you.

(e) Now try $(1.001)^{1000}$ and $(1.0001)^{10000}$. Toward what value do these numbers seem to converge?

Hint: Compare with $e \approx 2.71828 \dots$

(f) Why, for a rate $x > 0$ and an integer $n \geq 2$, does $(1 + x)^n > 1 + nx$ hold?

Hint: $(1 + x)^n = 1 + nx + \binom{n}{2}x^2 + \dots$. The term nx is linear growth. The later terms, such as $\binom{n}{2}x^2$, are the bonus from compounding.

For a fixed rate $x > 0$, $(1 + x)^n$ grows much faster than $1 + nx$ when n is large. The difference comes from the cross terms in the binomial expansion: interest itself earns interest. This is compound growth, the idea behind bank interest and population growth; with a negative rate, the same formula describes radioactive decay. When $x = 1/n$ and $n \rightarrow \infty$, $(1 + 1/n)^n$ converges (a result admitted here, not proved by these trials) to $e \approx 2.718$, one of the central constants of analysis.

Report 3: (a) $(1.01)^2 = 1 + 0.02 + 0.0001 = 1.0201$. (b) $(1.01)^{10} \approx 1.10462$, about 0.42% above the linear value 1.10. (c) $(1.01)^{100} \approx 2.7048$, about 35% above the linear value 2. (d) $(1.01)^{365} \approx 37.78$: a daily rate of 1% multiplies by more than 37 in one year. (e) $(1.001)^{1000} \approx 2.7169$ and $(1.0001)^{10000} \approx 2.71815$. They converge toward $e \approx 2.71828$. (f) For $x < 0$ and $n \geq 2$, the terms $\binom{n}{k} x^k$ are strictly positive, so $(1 + x)^n < 1 + nx$. For $x = 0$ or $n = 1$, equality holds.

FURTHER READING

- **Euler, L.** (1770), *Elements of Algebra*, trans. John Hewlett, 1822. *A historical route through elementary algebra; begin with Part I.*
- **Klein, F.**, *Elementary Mathematics from a Higher Standpoint, vol. I: Arithmetic, Algebra, Analysis*, trans. Gert Schubring, Springer, 2016. *A broader view of school mathematics, for a second reading with more experience.*
- **Courant, R. & Robbins, H.**, *What Is Mathematics?*, 2nd ed., revised by Ian Stewart, Oxford UP, 1996 (first published 1941). *A lively panorama; the opening chapters develop numbers and their operations.*